Chapter 2. Basic Asset Model

In the previous chapter we have seen a simple one-time consumption model. That was a model under certainty.

In this chapter we shall introduce a new model where uncertainty plays a role.

In place of consumption goods we now have n assets. Each asset is modeled by a random variable.

For simplicity we assume that all the assets (random variables) take a finite number of discrete values.

We consider a one-period economy. At the beginning of the period, a number of investors enter the market and freely trade the assets.

By the end of the period, payoff of all the investments will be made.

Suppose that by the end of the period ther
can be m different *scenarios* in the world.

The payoffs are given by a matrix $A = (a_{ij})_{m\times}$
where a_{ij} represents the payoff of a unit o
asset j if scenario i takes place.

For example,

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$$

tells the payoffs of two assets in a world wher
three scenarios may occur.

We may write $A = (a_1, ..., a_n)$ and assume tha
none of the columns is a zero vector. In som
cases, the assets are of *limited liability*, and s
$a_{ij} \geq 0$. But in general, this need not to b
the case.

A *portfolio* is a collection of assets. In th
case, it can be represented by an n-dimension
vector $w \in \Re^n$, where w_i is the *weight* of th
i-th asset.

In our model we assume that N investors enter the market and trade freely on the assets.

Each of the investor holds a given amount of budget, say b_i.

For each type of asset, the amount of supply is finite.

The key question that we are interested in is:

> *What will be the prices of the assets in the market?*

At a first glance, the answer is easy: A fair price of an asset should be its expected return (discounted to date).

There are a number of problems with this approach, however.

Problem No. 1:

In general, investors *do not* necessarily agre on the probability distribution of scenarios.

Problem No. 2:

Consider the contract of receiving a stock i a future date, say three months from nov Suppose that everybody agrees that the stoc price follows a known distribution (e.g. logno mal distribution). What is the price of such contract?

It is wrong to calculate the expected return a the delivery date, and set it as the price of th contract.

A correct price is simply S where S is the cu rent stock price. Because otherwise an *arb trage opportunity* would exist.

Problem No. 3:

Utility of each investor should matter as well.

Consider the *St. Petersburg paradox*: Suppose that one is invited to play the following game. If the first *head* occurs at the i-th toss of a fair coin, then one receives 2^i dollars. What is the "fair" price that one has to pay for entering such a game?

The expected payoff is infinite!

This means that the risk preferences of investors must also play a role in the prices.

In the rest of the course we will be concerned with how prices should/would be determined, taking into account of the utilities, no-arbitrage arguments, etc.

A One-Period Asset Investment Model

- The world exists for one period (0 to 1
  Invest at 0, payoff at 1
- There are n assets in the market with pay
  off matrix A
- Total amount of asset i is M_i
- There are N investors in the economy
- Investor k holds an initial budget of b_k

- Investor k's subjective believe of the probability distribution is q^k

- The cardinal utility function of investor k is U_k

- Frictionless market

- Short selling is allowed

Let the prices (value) of the assets at time 0 be $p \in \Re^n_{++}$.

The portfolio problem of investor k is

$$\begin{array}{ll} \max & (q^k)^T U_k(Aw) \\ \text{s.t.} & p^T w \le b_k \end{array}$$

Theorem 1 *If A is full-column rank and U*
is increasing and strongly concave, then th
above problem has a unique slack-free optim
solution.

Proof. Under the conditions we know tha
$f_k(w) = (q^k)^T U_k(Aw)$ is strongly concave.
strongly concave function always has bounde
upper-level sets. So the problem has an op
timal solution. The uniqueness and slack-fre
property follow from the monotonicity.

QE

Let the optimal solution be $w_k^*(p)$. In an equ
librium we must have

$$\sum_{k=1}^{N} w_k^*(p) = M$$

From this equation we can determine the pric
vector p.

However, one may criticize: *This formula is totally useless!*

Why? *How can one know the utility and the subjective probability distribution of each investor?*

Conclusion: Something more sensible should be done.

This will be the topic of our discussion in later chapters.

In order to understand the asset model better, we first study the payoff matrix A.

Some terminology:

A riskless portfolio w: $Aw = R\iota$ and $\iota^T w = 1$

(A riskless portfolio may not exist)

Riskless asset: An asset with payoffs $R\iota$.

Duplicable portfolio w_1: $\exists w_2$ such that $Aw_2 =$ Aw_1.

Redundant asset: Asset j is redundant if i payoff a_j is linearly dependent on other columns.

Dominant asset: Asset j is said to be dom nated by asset i if $a_i \geq a_j$.

Derivative asset: An asset whose payoff is de termined by the payoff of some other (prim tive) asset(s). (Forward contracts, options,..

Insurable states: State (Scenario) i is called insurable if there is η such that $A\eta = \iota_i$.

Such an portfolio is also known as an *Arrow-Debreu security*.

Example:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 2 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

In this case, asset 1 is riskless. Asset 4 is redundant since $a_4 = (a_2 + a_3)/2$. Portfolio $(0,1,1,0,0)^T$ is duplicable. Asset 3 dominates asset 4. Finally, state 1 is insurable since $A\eta = \iota_1$ with the Arrow-Debreu security $\eta = (-1,1,1,-2,1)^T$.

In the previous example, if the prices for th assets are: $p = (1.8, 1.6, 4.1, 2.3, 2)^T$ then or can construct the following portfolio: $w =$ $(1,0,-1,1,0)^T$. It is easy to see that:

$$Aw = \iota_4 \geq 0 \text{ and } p^T w = 0$$

So this will possibly bring in something an costs nothing!

This is what we call *an arbitrage opportunity*

Definition 1 *An arbitrage opportunity of th first type is w such that*

$$p^T w = 0 \textit{ and } Aw \geq (\neq) 0$$

Another way of forming a "clever" strategy is: $w = (0, -0.5, -0.5, 1, 0)$.

We see that

$$Aw = 0 \text{ and } p^T w = -0.55$$

This is again a free lunch! (Sell something and loose nothing)

Definition 2 *An arbitrage opportunity of the second type is w such that*

$$p^T w < 0 \text{ and } Aw \geq 0$$

If these kind of possibilities would exist in an economy (A, p), then the investors need only take this advantage to make an infinite wealth! Eventually, price will adjust and make arbitrage impossible.

A basic hypothesis in finance (Hypothesis 1 in Chapter 1) is that there should be no arbitrage opportunities.

Now we shall investigate the conditions unde which the above stated arbitrage will be im possible.

Definition 3 *A vector $q \in \Re^m$ is said to sup port the price p if*

$$A^T q = p$$

If q_i is the probability of event i to occur, the p_j can be interpreted as the expected payoff o the asset j.

In general there can be many supporting vec tors for a given price p. (We assume $p > 0$).

To check whether an economy (A, p) has a sup porting vector is easy. (*Why?*)

Theorem 2 *An economy (A, p) allows no second type arbitrage opportunity* iff *there exists a nonnegative supporting vector.*

To prove this theorem we first mention the well known Farkas Lemma:

Lemma 1 *(Farkas Lemma) The following two statements are equivalent:*

For any y, if $B^T y \geq 0$ then $b^T y \geq 0$

and

There is $x \geq 0$ such that $b = Bx$

Proof of Lemma 1: It is easy to see that if the second statement is true, then the first statement must be true. The difficult part of the lemma is to prove the reverse direction also holds.

Consider the following linear program

$$\begin{array}{ll} \max & 0^T x \\ \text{s.t.} & Bx = b,\ x \geq 0 \end{array}$$

The dual problem is:

$$\begin{array}{ll} \min & b^T y \\ \text{s.t.} & B^T y \geq 0 \end{array}$$

The dual has a feasible solution: $y = 0$.

So, by duality relation, the primal has a feasib solution iff the dual is bounded. This mear that:

the primal is feasible iff for y satisfying $B^T y \geq 0$ it must follow $b^T y \geq 0$.

This proves the Farkas Lemma.

QE

Proof of Theorem 2: Using Farkas Lemma: there is no arbitrage opportunity of the second type, i.e. there is no w such that $Aw \geq 0$ and $p^T w < 0$, implies that there is $q \geq 0$ such that $A^T q = p$, and vice versa.

QED

Is it easy to check whether an economy (A, p) allows no arbitrage of the second type?

Yes. This boils down to a linear programming feasibility problem.

Finally we will present a characterization of both arbitrage opportunities.

Theorem 3 *An economy (A, p) allows no first nor second type arbitrage opportunities* iff *there exists a positive supporting vector.*

To prove this assertion we need another lemm
about linear programming:

Lemma 2 *For any linear program*

$$\begin{array}{ll} \max & c^T x \\ \textit{s.t.} & Bx = b \\ & x \geq 0 \end{array}$$

and its dual problem

$$\begin{array}{ll} \min & b^T y \\ \textit{s.t.} & B^T y - s = c \\ & s \geq 0 \end{array}$$

if they have optimal solutions, then they mus have a pair of strictly *complementarity optim solutions, i.e. a pair of optimal solutions x^* an (y^*, s^*) such that*

$$x^* + s^* > 0$$

This result is a bit technical and we shall not prove it here. Interested reader can consult, e.g.,

Roos, Terlaky and Vial, *Theory and Algorithms for Linear Optimization*, John Wiley & Sons, 1997.

Proof of Theorem 3: It is easy to see that if there is a positive support q, then there can be no first nor second type arbitrage opportunities.

We now prove the other direction: Since there is no second type arbitrage opportunities, so

$$\begin{array}{ll} \text{max} & 0^T x \\ \text{s.t.} & A^T x = p,\, x \geq 0 \end{array}$$

and

$$\begin{array}{ll} \text{min} & p^T y \\ \text{s.t.} & Ay - s = 0,\, s \geq 0 \end{array}$$

must have optimal solutions.

Moreover, by Lemma 2, they must have *strict* *complementary* optimal solutions, i.e. $x^* \geq$ and $s^* \geq 0$ with $x^* + s^* > 0$.

Now, s^* cannot have any positive element due to the fact there is no first type arbitrag opportunities.

So, $s^* = 0$ and we must have $x^* > 0$.

QE

Checking the condition of Theorem 3 can b done by an *interior point method* for linear pro gramming.

From now on we assume

Assumption 1 *There is no first nor secon type arbitrage opportunities in the economy.*

Theorem 4 *If the assets are all of limited liability, i.e.* $a_{ij} \geq 0$ *for all* i *and* j*, then there is no first type arbitrage opportunity implies that there is no second type arbitrage opportunity.*

In general, however, it is possible that there is no first type arbitrage opportunity, while the second type arbitrage opportunity exists, and vice versa.

The proof of Theorem 4 and the above assertion will be left as a homework.

Remark that in defining arbitrage opportunities we do not assume that the investors agree on the probability distribution of the scenarios (states). But we do assume that all the investors agree on the payoff of each scenario.

There is a difference between *possibility* and *probability*.

Under Assumption 1, there is $q > 0$ such tha
$A^T q = p$. Let

$$R = 1/\iota^T q$$

and

$$q' = qR$$

we have

$$A^T q' = Rp$$

The vector q' is known as a *risk-neutral (*
martingale) probability.

The risk-neutral (martingale) distribution is i
general not unique.

Clearly, if there is no arbitrage, then any tw
portfolios (securities) having the same patter
of payoffs will have the same price. That is,

$$A(\Delta w) = 0 \Rightarrow p^T \Delta w = 0$$

This means that in an economy with no arbitrage opportunities, a portfolio with a certain payoff must be priced uniquely.

However, this does not imply that *any* pattern of payoff can be replicated by a portfolio in the economy. We will discuss this point later.

Consider an economy (A, p) with no arbitrage opportunities. Suppose that a new asset is introduced with payoff $a \in \Re^m$.

What can the price of the new asset be without introducing any arbitrage opportunities?

The interval of the price can be determined b
the following pair of linear programs:

$$\begin{array}{ll} \text{minimize} & a^T y \\ \text{subject to} & A^T y = p \\ & y \geq 0 \end{array}$$

and

$$\begin{array}{ll} \text{maximize} & a^T y \\ \text{subject to} & A^T y = p \\ & y \geq 0 \end{array}$$

Let the respective optimal values be v_l and v

Theorem 5 *For no arbitrage opportunity t exist: If $v_l = v_u$, then $p(a) = v_l = v_u$; If $v_l < v$ then $p(a) \in (v_l, v_u)$.*

Proof: In case $v_l = v_u$, any $p(a) \neq v_l = v_u$ will cause the new economy $((A, a), (p, p(a)))$ having no supporting vector at all, hence allowing arbitrage opportunities.

Consider $v_l < v_u$. Let y_l be an optimal solution to the first LP, and y_u be an optimal solution to second LP. Let $q > 0$ be a positive supporting vector. Clearly,

$$a^T q \in [v_l, v_u]$$

Now we either have

$p(a) \in (v_l, a^T q)$, or $p(a) = a^T q$, or $p(a) \in (a^T q, v_u)$

Without loss of generality, assume the first case. We have

$$p(a) = \lambda v_l + (1 - \lambda) a^T q$$

with $\lambda \in (0, 1)$. Then,

$$\bar{q} = \lambda y_l + (1 - \lambda) q$$

is the desired positive supporting vector. QED

The last part of the theorem is a little bit s
trange: *How can the price not be uniquely de*
termined? (Note: Any price in the interval
allowed, but it is not a variable!)

In an equilibrium, certainly the price will con
verge to a value in (v_l, v_u).

It is interesting to note the duals of the tw
LP problems. They are:

$$\begin{array}{ll} \text{maximize} & p^T x \\ \text{subject to} & Ax \leq a \end{array}$$

and

$$\begin{array}{ll} \text{minimize} & p^T x \\ \text{subject to} & Ax \geq a \end{array}$$

The first one wants to find the most "expen
sive" portfolio dominated by a and the sec
ond one is seeking for the "cheapest" portfoli
dominating a.

As there are many assets being traded in the market, it is plausible to assume that in the end there will be no freedom any more in the supporting vectors. That is why we now introduce the next important concept:

Definition 4 *An economy* (A, p) *is called complete if there is a unique* $q > 0$ *such that* $p = A^T q$.

Clearly, by Theorem 3 a complete economy should allow no arbitrage opportunities. Moreover, there is a unique risk-neutral (martingale) probability distribution. An immediate consequence is the following pricing result:

Corollary 1 *In a complete market, if a new asset is introduced with payoff vector* a*, then in the absence of arbitrage, its price must be* $q^T a$.

In the new economy the unique positive supporting vector remains unchanged.

Another way to interpret the complete marke
hypothesis is the following.

As we discussed before, in an arbitrage-free e
conomy any *replicating portfolio* has a uniqu
price. (A replicating portfolio is one whic
mimics a given pattern of payoffs). Howeve
we cannot guarantee that all types of payof
can be replicated.

Remember that an Arrow-Debreu security mir
ics a given state (scenario). So, if Arrow
Debreu securities exist for all states, then i
fact we can mimic *any* kind of payoffs.

In this sense, the market can be thought to b
complete.

What does this mean? This means in fact tha

$$\text{rank}(A) = m$$

As a consequence we see that in a complete market, there must be a unique supporting vector.

In a complete market, any state (scenario) ι_i ($i = 1, 2, ..., m$) can be replicated by an Arrow-Debreu security which has a unique price. This price is also called *the state price*.

Therefore, in a complete market a new asset (financial product) can be easily priced using state prices:

$$p(\text{[new-asset]}) = \sum_{i=1}^{m} \text{[new-asset]}_i \times p_{s_i}$$

where p_{s_i} stands for the i-th state price.

Note that we also called a state (scenario) *insurable* if there exists an Arrow-Debreu security. Hence, in a complete market, all states can be insured. But probably the costs for insurance will be different.

Finally, in the remainder of this chapter we w
discuss how to estimate the expected return o
an asset.

Note that the price and return of an asset
relative. So we may start by assuming eac
unit asset is scaled to price 1. Let A be scale
in this way too.

Let the true probability of the occurrence of
be π_i.

Define ω whose value is q_i/π_i in scenario i.

Now, $\iota = A^T q$, and this means that

$$1 = E(\omega z_j)$$

for all j where z_j is the return of asset j.

This can also be seen as

$$E(z_j) = \frac{1 - \text{Cov}(\omega, z_j)}{E(\omega)}$$

Now,

$$E(\omega) = \iota^T q := 1/R$$

Note that R represents the *riskless return*, because if A has a column with identical elements, then this element, which is by definition the riskless return, must equal R.

So

$$E(z_j) = R - R\text{Cov}(\omega, z_j)$$

So far this relation does not seem to be very useful. Especially, we do not know the meaning of ω. However, we will see a similar formula in the CAPM model later. In that model we will understand much better the relationship between the expected return and the covariance of the return and the market portfolio.