Chapter 4. Binomial Trees

According to what we have learned in Chapter 2, in the absence of arbitrage there must be a positive supporting vector in a market with finite assets and scenarios.

Recall that we have shown the prices of assets must obey the relationship

$$p = A^T q$$

where $q_j > 0$, $j = 1, ..., m$, and $\sum_{j=1}^{m} q_j = 1$.

In a sense, it looks like: *the prices of assets are determined in a risk-neutral way with probability distribution* q.

Important: This q is called risk-neutral probability and it may have nothing to do with *real* probability distribution of scenarios!

Consider now the problem of option pricing.

A European call option is the right to buy a
underlying asset, say a stock, with a prescribe
price at a prescribed date. Let the value o
such a call option be $c(S, K, t)$ where S is th
current price of the asset, K is the exercis
price and t is the time to maturity.

Symmetrically, a European put option is th
right to sell an underlying asset, say a stock
with a prescribed price at a prescribed date
Let its value be $p(S, K, t)$.

We call those assets, whose value depend o
the realization of some other assets, *derivativ*
assets. Examples of those include options, fu
tures, forward contracts and swaps.

Let us consider an extremely simply situation: We are in day $T-1$, one day away from maturity. The price of the stock is S. Suppose that we have two possible outcomes of the stock value for tomorrow: its value becomes either Su or Sd, where $u > 1$ and $0 < d < 1$.

Correspondingly, the value of the call option would become either $\max\{Su-K, 0\}$ or $\max\{Sd-K, 0\}$.

The question is: *What is the value of the option on day $T-1$?*

This can be considered as a world with three assets (the stock, the option and the riskless asset) and two scenarios.

The payoff matrix is:

$$A = \begin{bmatrix} Su, & (Su-K)_+, & e^{r\Delta t} \\ Sd, & (Sd-K)_+, & e^{r\Delta t} \end{bmatrix}$$

and the price vector of the assets is

$$p = \begin{bmatrix} S \\ c(S,K,T-1) \\ 1 \end{bmatrix}$$

where $c(S,K,T-1)$ is the unknown.

By Theorem 3 of Chapter 2, there should exis
a $q > 0$ vector such that $p = A^T q$. That is,

$$\begin{aligned} S &= Suq_1 + Sdq_2 \\ c(S,K,T-1) &= (Su-K)_+ q_1 + (Sd-K)_+ q \\ 1 &= e^{r\Delta t} q_1 + e^{r\Delta t} q_2 \end{aligned}$$

From the first the the third equation we derive

$$\begin{aligned} q_1 &= \frac{1 - de^{-r\Delta t}}{u - d} \\ q_2 &= \frac{ue^{-r\Delta t} - 1}{u - d} \end{aligned}$$

Consequently,

$$\begin{gathered} c(S, K, T-1) = [(Su - K)_+(1 - de^{-r\Delta t}) \\ +(Sd - K)_+(ue^{-r\Delta t} - 1)]/(u - d) \end{gathered}$$

What about u and d?

In reality, people may have good means to estimate the expected return rate, μ, of a stock, and the standard deviation on the rate, σ.

Let the real probability of the scenarios be π_1 and π_2.

Then we have

$$\begin{aligned} Se^{\mu\Delta t} &= Su\pi_1 + Sd\pi_2 \\ \sigma^2\Delta t &= u^2\pi_1 + d^2\pi_2 - (u\pi_1 + d\pi_2)^2 \\ 1 &= \pi_1 + \pi_2 \end{aligned}$$

The first and the last equation give

$$\pi_1 = \frac{e^{\mu\Delta t} - d}{u - d} \text{ and } \pi_2 = \frac{u - e^{\mu\Delta t}}{u - d}$$

Substituting this into the second equation yiel
s

$$\sigma^2\Delta t = (u + d)e^{\mu\Delta t} - ud - e^{2\mu\Delta t}$$

We now claim that

$$u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = e^{-\sigma\sqrt{\Delta t}}$$

approximately solve the equation.

This can be seen as

$$e^x \approx 1 + x + x^2/2$$

when x is small.

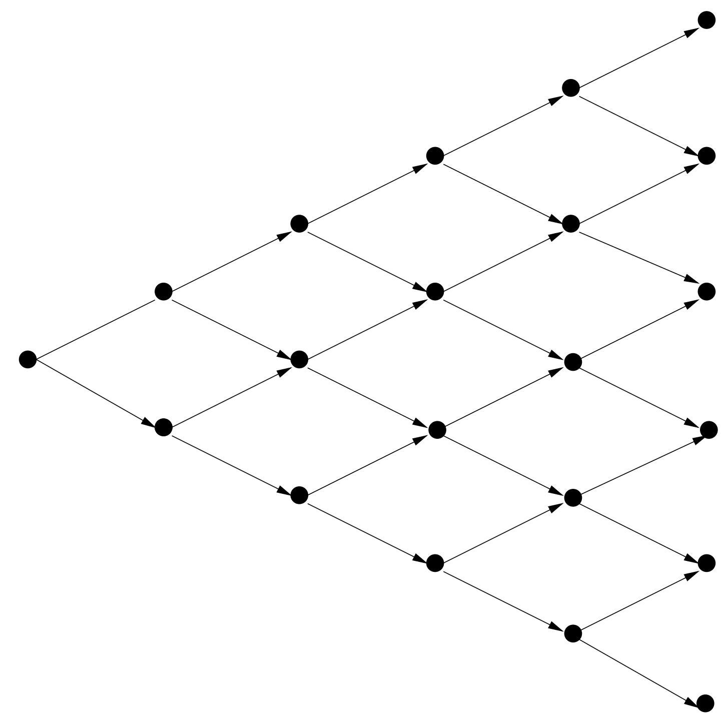
Therefore, one possible way to generate scenarios, fitting the variance is to let

$$u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = e^{-\sigma\sqrt{\Delta t}}$$

An interesting phenomenon is that the expected return rate μ does not play a role here.

Certainly this is an over-simplification of the reality. However, this idea can be used in a *multi-stage* way.

To save from the exponentially expanding number of nodes, we use the *recombining binomial tree* technique.



Let us consider a binomial tree with N stages.

Let the value of the option at stage i and level j be $f_{i,j}$. Note that the price of the underlying stock is

$$S_{i,j} = S_0 u^j d^{i-j}.$$

By recursion we have

$$\begin{aligned} f_{i,j} = [(&f_{i+1,j+1}(1 - de^{-r\Delta t}) \\ &+ f_{i+1,j}(ue^{-r\Delta t} - 1)]/(u-d) \end{aligned}$$

where $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$.

The terminal value of the option is known:

$$f_{N,j} = (S_0 u^j d^{N-j} - K)_+$$

The recursion is thus complete.

An American option allows to exercise the righ
at any time before the maturity.

However, it is never wise to early exercise a
American call option on a non-dividend payin
stock before the maturity.

It can be wise to exercise an American pu
option before the maturity.

To evaluate an American put option, one ca
use the same binomial tree technique. The or
ly difference is that now we consider the pos
sibility of early exercise:

$$\begin{aligned} f_{i,j} &= \max\{X - S_0 u^j d^{i-j}, \\ &\quad [(f_{i+1,j+1}(1 - de^{-r\Delta t}) \\ &\quad + f_{i+1,j}(ue^{-r\Delta t} - 1)]/(u-d)\}. \end{aligned}$$

Another issue in the option pricing problem is the stock dividend policy.

Usually dividend payment is known in advance, say D, at certain point in time, say τ.

One can thus think of a process S' defined as

$$S' = \begin{cases} S, & \text{when } i\Delta t > \tau \\ S - Dd^{-r(\tau - i\Delta t)}, & \text{when } i\Delta t \leq \tau \end{cases}$$

Intuitively, S' is the uncertain part of the stock price.

Therefore, a binomial tree can be constructed as before. Now the stock price is modeled on node (i, j) as

$$S'_0 u^i d^{i-j} + Dd^{-r(\tau - i\Delta t)}$$

when $i\Delta t < \tau$, and

$$S'_0 u^i d^{i-j}$$

when $i\Delta t \geq \tau$.