

Chapter 6. Mean-Variance Analysis

According to our discussion before, in an efficient market all the investors will finally have the same vision about the market. This implies, in particular, that they will share the same expectation of the return of the assets being traded.

For this and the next chapter, we will assume:

Assumption 1 *All the investors have the same vision about the market.*

According to the basic law of finance, there is no free lunch in the world; there are only trade-offs.

What we trade for our gains? *The risks!*

How to model the gains and the risks?

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Solution by Markowitz (1952): Gains can be modeled by the mathematical expectation of the return, and risks by its variance.

This is in agreement with the expected utility theory that we developed in Chapter 1, e.g., in the following two situations:

- If the utility function is quadratic, say

$$U(z) = z - \frac{c}{2}z^2$$

In this case,

$$E[U(z)] = E[z] - \frac{c}{2}(V(z) + (E[z])^2)$$

So indeed the expected utility function has to do with two parameters: $E[z]$ and $V(z)$

- If the asset return z obeys (multivariate) normal distribution

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From now on we will adopt the concept of Markowitz. Remark that the mathematical expectation of a random variable is also called the *mean*.

So we are facing an asset (or a portfolio) z and we measure its return by $E[z]$ and its risk by $V(z)$. Our utility function can then be written as a function of $E[z]$ and $V(z)$, i.e. $U(E[z], V(z))$.

Assumption 2 *Investors are nonsatiated, i.e. $U_1 > 0$.*

Assumption 3 *Investors are risk averse, i.e. $U_2 < 0$.*

Next we consider a setting when a portfolio has to be formed based on this utility function.

Suppose that n assets can be chosen to form a portfolio. Let them be $z_1, z_2, \dots, z_n$. For convenience their current prices are all scaled down to 1.

Let

$$d = (E[z_1], \dots, E[z_n])^T$$

and

$$\Theta = \text{Cov}(z, z) (= E[(z - d)(z - d)^T])$$

For convenience, assume that the budget available is 1.

What is the investor going to do?

He should solve the following problem:

$$\begin{aligned} \max \quad & U(d^T w, w^T \Theta w) \\ \text{s.t.} \quad & i^T w = 1 \end{aligned}$$

Now we introduce a *parametric* method for solving the problem.

Let

$$\mu = d^T w$$

(*the level of return*). Then the only solution that can be interesting to the investor is the solution of:

$$(P_\mu) \quad \begin{array}{ll} \min & \frac{1}{2} w^T \Theta w \\ \text{s.t.} & v^T w = 1 \\ & d^T w = \mu \end{array}$$

Let the optimal value of the above problem be $f(\mu)$. The portfolio optimization problem faced by the investor is simply one dimensional:

$$\max_{\mu \in \mathfrak{R}_+} U(\mu, 2f(\mu))$$

The above analysis shows that to solve the portfolio problem we need only to solve (P_μ) for all $\mu \geq 0$.

In fact this is more than what we need: The investment problems of *all* investors of the similar type can be solved this way, in the sense that they can all be reduced to a simple one dimensional optimization problem.

That is why we will now concentrate on the problem (P_μ) .

In the term of mathematical programming, this is a quadratic programming problem.

Its financial interpretation is: *How can one minimally diversify risks while maintaining the expected return at a certain level?*

We assume that there is no redundant asset.

For technical reasons, we also assume for the moment that there is no riskless asset in the model. A riskless asset is so special that it will be treated separately later on.

Under these two conditions, we know that the covariance matrix Θ is positive definite.

We consider (P_μ) again. Define its Lagrangian to be

$$L(w; \lambda, \gamma) = \frac{1}{2} w^T \Theta w + \lambda(1 - \iota^T w) + \gamma(\mu - d^T w)$$

The K-K-T optimality condition states that an optimal solution w^* must satisfy the following condition:

$$\left\{ \begin{array}{lcl} \frac{\partial L}{\partial w} |_{w=w^*} & = & 0 \\ \iota^T w^* & = & 1 \\ d^T w^* & = & \mu \end{array} \right.$$

From this system of equations we can solve out:

$$w^* = \lambda \Theta^{-1} \iota + \gamma \Theta^{-1} d$$

where

$$\lambda = \frac{C - \mu B}{\Delta} \text{ and } \gamma = \frac{\mu A - B}{\Delta}$$

and

$$\begin{aligned} A &= \iota^T \Theta^{-1} \iota \\ B &= \iota^T \Theta^{-1} d \\ C &= d^T \Theta^{-1} d \\ \Delta &= AC - B^2 \end{aligned}$$

Remark that $\Delta > 0$ due to the Cauchy-Schwarz inequality.

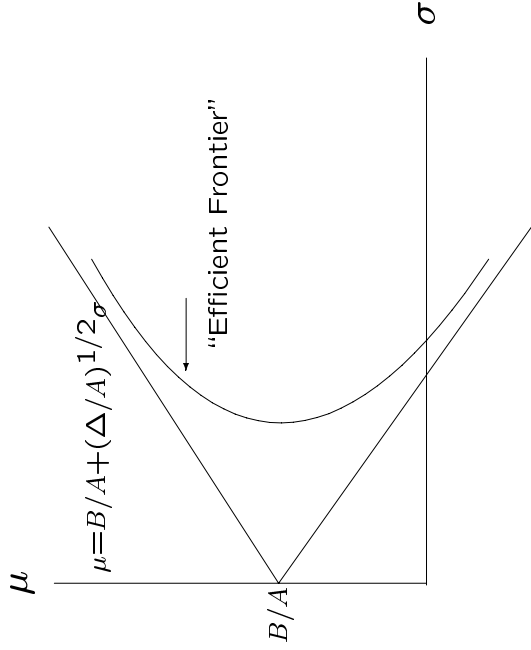
The variance of the optimal portfolio is

$$\begin{aligned} \sigma^2 &= (w^*)^T \Theta w^* \\ &= \frac{A\mu^2 - 2B\mu + C}{\Delta} \end{aligned}$$

This shows that

$$f(\mu) = \frac{A\mu^2 - 2B\mu + C}{2\Delta}$$

In the space of μ and σ this is a *hyperbola*:



The quantity σ is also known as *the standard deviation*.

The global minimum standard deviation is

$$\sigma = 1/\sqrt{A}$$

with the corresponding mean $\mu = B/A$.

The slopes of the asymptotes can be computed too:

$$\begin{aligned} \frac{d\mu}{d\sigma} &= \frac{d\mu}{d\sigma^2} \frac{d\sigma^2}{d\sigma} \\ &= \frac{\Delta}{2A\mu - 2B} 2\sigma \\ &= \frac{\sqrt{\Delta}}{A\mu - B} \sqrt{A\mu^2 - 2B\mu + C} \end{aligned}$$

and so

$$\lim_{\mu \rightarrow \pm\infty} \frac{d\mu}{d\sigma} = \pm \sqrt{\frac{\Delta}{A}}$$

The upper part of the hyperbola is called *the efficient frontier* of the mean-variance portfolios.

The efficient frontier tells us about the necessary trade-offs between risks and returns.

Each point on the efficient frontier corresponds to an optimal portfolio. A rational investor (satisfying our assumptions 2 and 3) will only choose a portfolio on the efficient frontier. His particular utility will determine which point on the efficient frontier to choose (people may have different appetite for risks).

As a typical case we assume that the mean of the global minimum standard deviation portfolio is positive, i.e. $B/A > 0$, and so $B > 0$.

For any $\mu \geq B/A$, the optimal (efficient) portfolio is:

$$w^*(\mu) = \frac{A(C - \mu B)}{\Delta} [\Theta^{-1} \iota / A] + \frac{B(\mu A - B)}{\Delta} [\Theta^{-1} d / B]$$

Notice that

$$\Theta^{-1} \iota / A$$

corresponds to the global minimum standard deviation portfolio, and

$$\Theta^{-1} d / B$$

corresponds to the efficient portfolio with $\mu = C/B (> B/A)$.

Let us call them w_g and w_f respectively.

We see that $w^*(\mu)$ is always a linear combination of w_g and w_f . Moreover, the coefficient add up to 1, namely,

$$\frac{A(C - \mu B)}{\Delta} + \frac{B(\mu A - B)}{\Delta} = 1$$

for all μ .

This is an interesting relation. It says basically that you need only to buy (or sell) these two portfolios to make your own efficient portfolio.

Let the random return generated by portfolio w be $z(w)$.

A peculiar fact is that the covariance between $z(w_g)$ and any other portfolio is a constant:

$$\text{Cov}(z(w_g), z(w)) = w_g^T \Theta w = \frac{e^T \Theta^{-1} \Theta w}{A} = \frac{1}{A}$$

For any given a and b , the portfolios

$$w_a = (1 - a)w_g + aw_f$$

and

$$w_b = (1 - b)w_g + bw_f$$

have the covariance

$$\begin{aligned} \text{Cov}(z(w_a), z(w_b)) &= w_a^T \Theta w_b \\ &= \frac{1}{A} + \frac{ab\Delta}{AB^2} \end{aligned}$$

This means that for fixed $a > 0$, the covariance can take any value if we let b vary.

This also shows that for $a > 0$ we may always choose

$$b = -\frac{B^2}{a\Delta}$$

to make a (non-efficient) portfolio, which has zero covariance with w_a .

We call w_b the *conjugate portfolio* w.r.t. w_a . As a matter of notation we write the conjugate portfolio of w_a to be

$$w_{c(a)}$$

Now we come to the point: *How to incorporate a riskless asset?*

Let the return on the riskless asset be R . The amount invested in the riskless asset will be

$$1 - \iota^T w$$

(Why shall we not invest all the wealth?).

So the total expected return will be

$$d^T w + R(1 - \iota^T w)$$

Let the expected return be parameterized by μ . Then we have

$$(d - R\iota)^T w = \mu - R$$

Then our portfolio problem becomes

$$\begin{aligned} (P_\mu)' \quad & \min \quad \frac{1}{2} w^T \Theta w \\ \text{s.t.} \quad & (d - R\iota)^T w = \mu - R \end{aligned}$$

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Similar to our derivation before, $(P_\mu)'$ can be solved by means of the K-K-T condition.

This time, the solution of the problem is:

$$w^*(\mu) = \xi \Theta^{-1} (d - R\iota)$$

with

$$\xi = \frac{\mu - R}{C - 2RB + R^2 A}$$

So, the corresponding optimal standard deviation is:

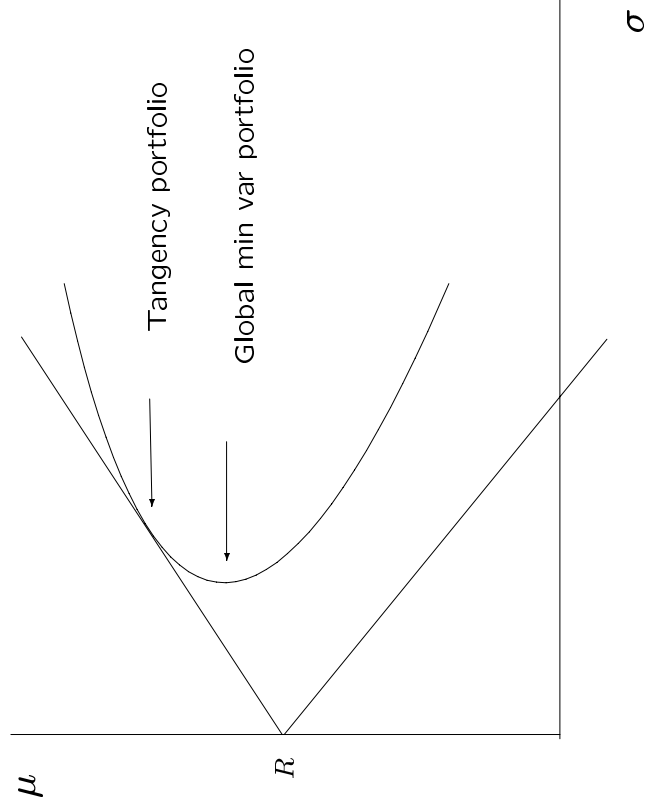
$$\sigma^2 = \frac{(\mu - R)^2}{C - 2RB + R^2 A}$$

In the space of (σ, μ) , this is a degenerate hyperbola. It consists of two rays starting from $(0, R)$.

In the next picture we put the new hyperbola and the “old” one together.

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$$\mu = R + \sigma(C - 2RB + R^2A)^{1/2}$$



In this picture there are two portfolios that de serve our special attention: One portfolio consists only of riskless assets and the second is portfolio consists only of risky assets. In symbols, they are:

$$w_0 = 0$$

and

$$w_t = \frac{\Theta^{-1}(d - R_L)}{B - AR}$$

The second portfolio is called the *tangency portfolio*.

The mean of the tangency portfolio is

$$\mu_t = \bar{d}^T w_t = \frac{C - BR}{B - AR}$$

and the variance

$$\sigma_t^2 = w_t^T \Theta w_t = \frac{C - 2RB + R^2A}{(B - AR)^2}$$

It is clear again that all optimal (efficient) portfolios, in the presence of a riskless asset, is always a linear combination of these two portfolios.

Proposition 1 *Either the inequalities*

$$\mu_t > B/A > R$$

or the reverse inequalities

$$\mu_t < B/A < R$$

holds.

Proof: We need to notice the following identities:

$$\begin{aligned} (\mu_t - \frac{B}{A})(\frac{B}{A} - R) &= (\frac{C - BR}{B - AR} - \frac{B}{A})(\frac{B - AR}{A}) \\ &= \frac{C - BR}{A} - \frac{B(B - AR)}{A^2} \\ &= \frac{CA - B^2}{A^2} = \frac{\Delta}{A^2} > 0 \end{aligned}$$

QED

Conclusions:

- The mean-variance analysis shows that the expected utility of a rational investor is related only to the mean and variance of the return, then one can analyze the tradeoff between the “return” and the risk using a simple curve: The efficient frontier of a quadratic program.
- As a conclusion of this analysis, any optimal portfolio is a combination of any two other optimal portfolios belonging to the efficient frontier.
- One may replace the variance by *down side deviation*. But the model becomes difficult to analyze.