Name:

MATH 520 Linear Algebra
Fall 2002-2003
Final Examination
8.1.2003

There are 10 questions with equal weight, and the weights add up to 40. Prove or disprove means: either you choose to give a proof or you give a counterexample. Please answer the questions in the space provided.

1. Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

2. Prove or disprove:

There exists a basis (p_0, p_1, p_2, p_3) of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

3. Give an example of a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ such that

$$f(av) = af(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but f is linear.

4. T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove or disprove:

> T is surjective.

5. Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove or disprove

$$\text{null}\,(T - \lambda I) \text{ is invariant under } S \text{ for every } \lambda \in \mathbf{F}.$$

6. Suppose that $T \in \mathcal{L}(V)$ is invertible and $\lambda \in \mathbf{F} \setminus \{0\}$. Prove or disprove

$$\lambda \text{ is an eigenvalue of } T \text{ if and only if } \tfrac{1}{\lambda} \text{ is an eigenvalue of } T^{-1}.$$

7. Prove or disprove:

There is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x_1, x_2)\| = |x_1| + |x_2|$$

for all $(x_1, x_2) \in \mathbf{R}^2$.

8. Suppose that $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove or disprove

$$U \text{ is invariant under } T \text{ if and only if } P_U T P_U = T P_U.$$

9. Prove or disprove:

If $T \in \mathcal{L}(V)$ is normal, then

$$\operatorname{range} T = \operatorname{range} T^*.$$

10. Does there exist a self-adjoint operator $T \in \mathcal{L}(\mathbf{R}^3)$ such that $T(1,2,3) = (0,0,0)$ and $T(2,5,7) = (2,5,7)$?