Name:

IE 513 Linear Programming
Fall 2000
Final Exam

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- Don't panic! The exam is easier than it appears.
- The answers are usually two to three lines.
- With the exception of Question 5, you need only use the information given in the question to come up with the answer.
- If you use a result from some source to solve a question, please give a proper reference.
- The exam weights add up to 50, with the weight of each question indicated on top.
- Good luck and have a good break!

1[10 points]. The following is a fundamental theorem of linear inequalities:

Theorem 0.1 *Assume that the system*

$$a_i x \leq \alpha_i \; i = 1, \ldots, m$$

is consistent. An inequality $a_0 x \leq \alpha_0$ *is a consequence of this system if and only if there exist non-negative real numbers* $\lambda_1, \ldots, \lambda_m$ *such that*

$$\sum_{i=1}^{m} \lambda_i a_i = a_0, \text{ and } \sum_{i=1}^{m} \lambda_i \alpha_i \leq \alpha_0.$$

(An inequality $a_0 x \leq \alpha_0$ is said to be a consequence of the system

$$a_i x \leq \alpha_i \; i = 1, \ldots, m$$

if it is satisfied by every x that satisfies the system.) Using the above result, prove the following

Corollary 0.1 *(Farkas' Lemma). An inequality* $a_0 x \leq 0$ *is a consequence of the system*

$$a_i x \leq 0 \; i = 1, \ldots, m$$

system if and only if there exist non-negative real numbers $\lambda_1, \ldots, \lambda_m$ *such that*

$$\sum_{i=1}^{m} \lambda_i a_i = a_0.$$

Solution. The zero vector satisfies $a_i x \leq 0$ for $i = 1, \ldots, m$. Therefore the hypothesis of the theorem is satisfied (consistency). Just apply the theorem.

2. [5 points] (The skew-symmetric model). Consider the following linear program

$$(SP)\min\{q^T x : Mx \geq -q, x \geq 0\},$$

where the matrix M is of size $n \times n$ and skew-symmetric ($M^T = -M$), and the vectors q and x are in $\Re^n$, with $q \geq 0$. This problem has a trivial optimal solution. What is it? (Hint: Consider 0.)

Solution. The zero vector is an optimal solution. Why? Because $x = 0$ is feasible since $q \geq 0$. Now, let x be an arbitrary feasible solution. Then $x \geq 0$ implies $q^T x \geq 0$. Thus it follows that zero is the optimal value of (SP), and $x = 0$ is an optimal solution.

3. [10 points] In reference to Question 2, associate with any vector x the vector

$$s(x) = Mx + q.$$

Show that $x \geq 0$ is optimal if and only if $s(x) \geq 0$ and $x^T s(x) = 0$. (Hint: use the result of the previous question, the skew-symmetry of M and the observation $z^T M z = (z^T M z)^T = z^T M^T z$.)

Solution. The zero vector is an optimal solution, but it does not have to be the unique optimal solution. Hence, our interest in this question.

Now, using the hint we have

$$z^T M z = (z^T M z)^T = z^T M^T z = -z^T M z$$

(last equality follows because $M^T = -M$) but this implies that $z^T M z = 0$ (why? because we have $z^T M z = -z^T M z$, the only such number we know is zero.)

Using this, we may write

$$q^T x = x^T (s(x) - Mx) = x^T s(x) - x^T M x = x^T s(x)$$

but since $q^T x = 0$ for any optimal x, the result follows.

4. [10 points] Consider the problems

$$\max\{\sum_{j=1}^{n} c_j x_j : \sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, \ldots, m, l_j \leq x_j \leq u_j, j = 1, \ldots, n.\} \tag{1}$$

and

$$\max\{\sum_{j=1}^{n} c_j x_j : \sum_{j=1}^{n} a_{ij} x_j = b_i, i \in I, l_j \leq x_j \leq u_j, j = 1, \ldots, n.\} \tag{2}$$

The following result is given:

Theorem 0.2 *If (1) has a feasible solution, then some set I of subscripts $1, 2, \ldots, m$ has the following properties: (i) problems (1) and (2) have precisely the same set of feasible solutions and (ii) problem (2) has a basic feasible solution.*

Using the above result, prove

Theorem 0.3 *If a system of m linear equations has a non-negative solution, then it has a solution with at most m positive variables.*

(Hint: Consider the system $\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, \ldots, m$ and $x_j \geq 0, j = 1, \ldots, n.$)

<u>Solution.</u> If the system $\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, \ldots, m$ and $x_j \geq 0, j = 1, \ldots, n.$ has a solution, then by the first theorem of the problem, there is some set I of subscripts $1, 2, \ldots, m$ such that this system and $\sum_{j=1}^{n} a_{ij} x_j = b_i, i \in I$ and $x_j \geq 0, j = 1, \ldots, n$ have precisely the same set of solutions and the second system has a basic feasible solution $x_1^*, x_2^*, \ldots, x_n^*$ ¿ Now, it only remains to be observed that $x_j^* = 0$ whenever variable x_j is nonbasic and that at most $|I|$ variables $(|I| \leq m)$ x_j are basic.

5. [5 points] Consider the following problem:

$$\begin{array}{lrcl} \max & -x_1 - 2x_2 & & \\ \text{s.t.} & -2x_1 + x_2 & \leq & 2 \\ & -x_1 + 2x_2 & \leq & 7 \\ & x_1 & \leq & 3 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Find, without using the simplex method, the optimal solution to the primal. Infer using duality and complementary slackness properties the dual optimal solution.

Solution. By inspection, or using graphics in two dimensions, the optimal solution to the primal is the origin in two dimensions. Then by complementarity, the origin in three dimensions is an optimal solution.

6. [10 points] Consider the following version of Farkas Lemma:

Theorem 0.4 *Let A be a matrix and let b be a vector. Then there exists a vector $x \geq 0$ with $Ax = b$, if and only if $b^T y \geq 0$ for each vector y with $y^T A \geq 0$.*

Use the above result to prove

Theorem 0.5 *Let A be a matrix and b be a vector. Then the system $Ax \leq b$ of linear inequalities has a solution x, if and only if $y^T b \geq 0$ for eqch $y \geq 0$ with $A^T y = 0$*

(Hint: $Ax \leq b$ has a solution x if $[I \ A \ -A]x' = b$ has a non-negative solution x'. Why? Convince me why, and then proceed.)

and

Theorem 0.6 *Let A be a matrix and b be a vector. Then the system $Ax \leq b$ of linear inequalities has a solution $x \geq 0$, if and only if $y^T b \geq 0$ for each $y \geq 0$ with $A^T y \geq 0$.*

(Hint: $Ax \leq b$ has a non-negative solution x if $[I \ A]x' = b$ has a non-negative solution x'. Convince me why this is true and proceed.)

Solution.

I. Let A' be the matrix $[I \ A \ -A]$, then $Ax \leq b$ has a solution x, if and only if $A'x' = b$ has a non-negative solution x'. Then apply the result of the given Farkas lemma to the latter system.

II. Let A' be the matrix $[I \ A]$, then $Ax \leq b$ has a solution $x \geq 0$, if and only if $A'x' = b$ has a non-negative solution x'. Then apply the result of the given Farkas lemma to the latter system.