IE 444 Operations Research in Finance
Spring 2005
Homework II, due: 24.2.2005

This is a purely theoretical homework. You can again work in groups of two. If you happen to find answers to these questions in a book or elsewhere, make sure to give a proper reference.

1. Consider an experiment where the set of possible outcomes is $\{1, 2, \ldots, m\}$, and suppose that n wagers concerning this experiment are available. If the amount x is bet on wager i then $xr_i(j)$ is receievd if the outcome of the experiment is j ($j = 1, \ldots, m$). In other words, $r_i(.)$ is the return function for a unit bet on wager i. The amount bet on a wager is allowed to be positive, negative or zero.

A betting strategy is a vector $x = (x_1, x_2, \ldots, x_n)^T$ with the interpretation that x_1 is bet on wager 1, x_2 on wager 2, and so on. If the outcome of the experiment is j, then the return from the betting strategy x is given by $\sum_{i=1}^{n} x_i r_i(j)$. Using duality theory of linear programming, establish whether the following assertion is true or false:

Exactly one of the following is true: Either

(a) There is a probability vector $p = (p_1, \ldots, p_m)^T$ for which

$$\sum_{j=1}^{m} p_j r_i(j) = 0 \text{ for all } i = 1, \ldots, n,$$

or else

(b) there is a betting strategy $x = (x_1, x_2, \ldots, x_n)^T$ for which

$$\sum_{i=1}^{n} x_i r_i(j) > 0, \text{ for all } j = 1, \ldots, m.$$

2. In some situations, the only type of wagers allowed are ones that choose one of the outcomes i ($i = 1, \ldots, m$) and then bet that i is the outcome of the experiment. The return from such a bet is often quoted in terms of "*odds*". If the odds against outcome i are o_i (often expressed as "o_i to 1"), then a one-unit bet will return either o_i if i is the outcome of the experiment or -1 if i is not the outcome. That is, a one-unit bet on i will either win o_i or lose 1.

(a) Give an expression for the return function $r_i(j)$.

(b) Suppose that the odds $o_1, o_2, \ldots, o_m$ are quoted. Using the result of the previous exercise find a condition in terms of o_i's for there not to be an arbitrage. When is a sure win possible?

(c) Based on your answer to part (b), assume you are faced with an experiment with three possible outcomes with the respective odds $1, 2, 3$. Is a sure win possible? If your answer is affirmative, what is a betting strategy that results in a sure win?

(d) **Bonus question**: In part (c), can you give a closed-form formula for a sure-win betting strategy?