IE 303.1 Modeling and Methods in Optimization
Fall 2008
HW2, due date: 9.10.2008

You are expected to work individually on the problems below. Have a good break!

1.[40 points] A production line consists of an ordered sequence of n production stages, and each stage has a manufacturing operation followed by a potential inspection. The product enters stage 1 of the production line in batches of size $B \geq 1$. As the items within a batchmove through the manufacturing stages, the operations might introduce defects. The probability of producing a defect at stage i is α_i. We assume that all the defects are non-repairable, so we must scrap any defective item. After each stage, we can either inspect all the items or none of them (we do not sample the items); we assume that inspection identifies every defective item. The production line must end with an inspection station so that we do not ship out any defective units. Our decision problem is to find an optimal inspection plan that specifies at which stages we should inspect the items so that we minimize the total cost of production and inspection. Using fewer inspection stations might decrease the inspection cost, but will increase the production costs because we might perform unnecessary manufacturing operations on some units that are already defective. The optimal number of stations will achieve an appropriate trade-off between these two conflicting cost considerations.

Suppose that the following data are available: (1) p_i, the manufacturing cost per unit at stage i; (2) f_{ij} the fixed cost of inspecting a batch after stage j, given that we last inspected the batch after stage i; (3) g_{ij}, the variable cost per unit for inspecting an item after stage j, given that we last inspected the batch after stage i. The inspection costs at station j depend on where the batch was inspected last, say at station i, because the inspector needs to look for defects incurred at intermediate stages $i+1, i+2, \ldots, j$.

a. Show that the inspection problem can be formulated as a shortest path problem on an appropriate network. Describe the network, i.e., what are the nodes and what are the arcs? Draw your network. (**Hint:** What is the meaning in terms of the original problem of sending one unit of flow from node i to some other node j?)

b. What is the cost c_{ij} per unit of going from node i to node j? Give a formula for c_{ij}.

c. For $n = 4$, pick your values for p_i, f_{ij}, g_{ij}, and solve your model in XPress-MP, and describe your optimal inspection plan. Attach an output file.

2.[40 points] A company has r residents $R_1, R_2, \ldots, R_r$, q clubs $C_1, C_2, \ldots, C_q$; and p political parties $P_1, P_2, \ldots, P_p$. Each resident is a member of at least one club and can belong to exactly one political party. Each club must nominate one of its members to represent it on the town's governing council so that the number of council members belonging to political party P_k is at most u_k. Is it possible to find a council that satisfies the "balancing" property? Find an appropriate network on which this problem can be formulated as a maximum flow problem. Show that if the maximum flow value in this network is equal to q, then the town has a balanced council; otherwise it does not. Illustrate your network with $r = 7$, $q = 4$, and $p = 3$.

3.[20 points] The following result is due to Kruskal and Hoffman [1956]:

An integral matrix A is totally unimodular if and only if for all integral vectors b and c both sides of the linear programming duality equation:

$$\max_x \{c^T x | x \geq 0, Ax \leq b\} = \min_y \{b^T y | y \geq 0, y^T A \geq c\} \quad (1)$$

are achieved by integral vectors.

Now, let $G = (V, E)$ be an undirected graph, and let M be the $|V| \times |E|$ incidence matrix of G (i.e., M is a $\{0, 1\}$-matrix with rows indexed by the nodes, and columns indexed by the edges of G, respectively, where $M_{v,e} = 1$ if and only if $v \in e$. For example for the triangle graph $\triangle$ we have the incidence matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is well-known that

M is totally unimodular if and only if G is bipartite,

(A graph $G = (V, E)$ is bipartite if V can be partitioned into two classes V_1 and V_2 such that each edge of G contains a node in V_1 and a node in V_2, example: graphs that arise in the transportation and assignment problems.)

Assume that each node is contained in at least one edge.

In the light of the above, give a "graph" interpretation of the following duality result: (let $\mathbf{1}$ represent a vector with all components equal to one)

$$\max_y \{\mathbf{1}^T y | y \geq 0, M^T y \leq \mathbf{1}, y \text{ integral}\} = \min_x \{\mathbf{1}^T x | x \geq 0, Mx \geq \mathbf{1}, x \text{ integral}\}. \quad (2)$$

I.e., what do the problems represent in the graph, and what does the duality result say? You will need the following definition in your answer: A *coclique* in a graph $G = (V, E)$ is a set of pairwise non-adjacent (not directly connected by an edge) nodes of G.

The duality equality (2) is known as the König Covering Theorem from 1932, after the Hungarian mathematician Dénes König. It is also known as the König-Rado edge cover theorem.