MATH 520 Linear Algebra
Fall 2005
HW III
6.10.2005 , due: 13.10.2005

An affine map T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a map of the form $Tx = Sx + a$, where a is a fixed vector in $\mathbf{R}^m$ and S is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$.

1. Prove or disprove: A map T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is affine if and only if for each $\lambda \in \mathbf{R}$ and each pair of points x_1 and x_2 in $\mathbf{R}^n$

$$T(\lambda x_2 + (1-\lambda)x_1) = \lambda T x_2 + (1-\lambda) T x_1.$$

A subset C of $\mathbf{R}^n$ is said to be *convex* if $(1-\lambda)x + \lambda y \in C$ whenever $x \in C$, $y \in C$ and $0 < \lambda < 1$.

2. Let TC denote the image of a convex set C under a map T. Prove or disprove: TC is convex if T is an affine map.

A subset K of $\mathbf{R}^n$ is said to be a cone if it is closed under positive scalar multiplication, i.e., $\lambda x \in K$ whenever $x \in K$ and $\lambda > 0$. It is called a *convex cone* if it is also convex.

3. Prove or disprove: A subset K of $\mathbf{R}^n$ is a convex cone if and only if it is closed under addition and positive scalar multiplication.

If A is $m \times n$ matrix, then define

$$Im(L_A) = \{Ax | x \in \mathbf{R}^n\}.$$

$Im(L_A)$ is a subspace of $\mathbf{R}^n$ spanned by the columns A. The dimension of $Im(L_A)$ is called the *column rank* of A and is denoted $r_c(A)$. Let B be a $n \times p$ matrix and consider the linear maps $L_A v = Av$ and $L_{AB} v = ABv$ for $v \in \mathbf{R}^n$. The dimension of $Im(L_{AB})$ is denoted $r_c(AB)$.

4. Prove or disprove: $r_c(AB) \leq r_c(A)$ and $r_c(AB) \leq r_c(B)$.