15.053 Tuesday, March 5

- **Duality**
  - The art of obtaining bounds
  - weak and strong duality
- Handouts: Lecture Notes

Bounds

- One of the great contributions of optimization theory (and math programming) is the providing of upper bounds for maximization problems
- We can prove that solutions are optimal
- For other problems, we can bound the distance from optimality

A 4-variable linear program

David has minerals that he will mix together and sell for profit. The minerals all contain some gold content, and he wants to ensure that the mixture has 3% gold, and each bag will weigh 1 kilogram.

Mineral 1: 2% gold, $3 profit/kilo
Mineral 2: 3% gold, $4 profit/kilo
Mineral 3: 4% gold, $6 profit/kilo
Mineral 4: 5% gold, $8 profit/kilo

maximize $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$$x_1 + x_2 + x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

A 4-variable linear program

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subject to

$$x_1 + x_2 + x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

-z	x_1	x_2	x_3	x_4		
1	3	4	6	8	=	0
0	1	1	1	1	=	1
0	2	3	4	5	=	3

Obtaining a Bound

-z	x_1	x_2	x_3	x_4		
1	-5	-4	-2	0	=	-8
0	1	1	1	1	=	1
0	2	3	4	5	=	3

Subtract 8 times constraint 1 from the objective function.

$-z - 5x_1 - 4x_2 - 2x_3 = -8$ $z + 5x_1 + 4x_2 + 2x_3 = 8$

Does this show that $z \leq 8$? YES!

Obtaining a Second Bound: Treat the operation as pricing out

-z	x_1	x_2	x_3	x_4			Prices
1	-2	-2	-1	0	=	-6	
0	1	1	1	1	=	1	3
0	2	3	4	5	=	3	1

Subtract 3 * constraint 1 and subtract constraint 2 from the objective function.

$-z - 2x_1 - 2x_2 - 1x_3 = -6$	$z + 2x_1 + 2x_2 + 1x_3 = 6$
Thus $z \leq 6$!	Which bound is better: 6 or 8?

Obtaining the Best Bound: Formulate the problem as an LP

-z	x_1	x_2	x_3	x_4		Prices
1	A	B	C	D	=	$-y_1-3y_2$
0	1	1	1	1	=	1	y_1
0	2	3	4	5	=	3	y_2

A: $3 - y_1 - 2y_2 \le 0$ ➔ $y_1 + 2y_2 \ge 3$

B: $4 - y_1 - 3y_2 \le 0$ ➔ $y_1 + 3y_2 \ge 4$

C: $6 - y_1 - 4y_2 \le 0$ ➔ $y_1 + 4y_2 \ge 6$

D: $8 - y_1 - 5y_2 \le 0$ ➔ $y_1 + 5y_2 \ge 8$

minimize $y_1 + 3y_2$

The problem that we formed is called the <u>dual problem</u>

minimize $y_1 + 3y_2$

Subject to $y_1 + 2y_2 \ge 3$

$y_1 + 3y_2 \ge 4$

$y_1 + 4y_2 \ge 6$

$y_1 + 5y_2 \ge 8$

y_1 and y_2 are unconstrained in sign

Summary of previous slides

- **If "reduced costs" are non-positive then we have an upper bound on the objective value**
- **The problem of finding the least upper bound is a linear program, and is referred to as the dual of the original linear program.**
- **To do: express duality in general notation.**
- **To do: show that the shadow prices solve the dual, and the bound is the optimal solution to the original problem**

<u>PRIMAL PROBLEM:</u>

<u>**maximize**</u> $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$$x_1 + x_2 + x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$$
$$x_1, x_2, x_3, x_4 \ge 0$$

<u>DUAL PROBLEM:</u>

minimize	$y_1 + 3y_2$
Subject to	$y_1 + 2y_2 \ge 3$
	$y_1 + 3y_2 \ge 4$
	$y_1 + 4y_2 \ge 6$
	$y_1 + 5y_2 \ge 8$

Observation 1.

The constraint matrix in the primal is the transpose of the constraint matrix in the dual.

Observation 2.

The RHS coefficients in the primal become the cost coefficients in the dual.

<u>PRIMAL PROBLEM:</u>

<u>**maximize**</u> $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$$x_1 + x_2 + x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$$
$$x_1, x_2, x_3, x_4 \ge 0$$

<u>DUAL PROBLEM:</u>

minimize	$y_1 + 3y_2$
Subject to	$y_1 + 2y_2 \ge 3$
	$y_1 + 3y_2 \ge 4$
	$y_1 + 4y_2 \ge 6$
	$y_1 + 5y_2 \ge 8$

Observation 3. The cost coefficients in the primal become the RHS coefficients in the dual.

Observation 4. The primal (in this case) is a max problem with equality constraints and non-negative variables

The dual (in this case) is a minimization problem with ≥ constraints and variables unconstrained in sign.

<u>PRIMAL PROBLEM (in standard form):</u>

<u>**max**</u> $z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$

s.t.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m$$
$$x_j \ge 0 \text{ for } j = 1 \text{ to } n.$$

<u>DUAL PROBLEM:</u>

<u>**min**</u> **v = ???**

s.t. ???

What is the dual problem in terms of the notation given above?

PRIMAL PROBLEM (in standard form):

$$
\begin{array}{llcl}
\textbf{max} & z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n & & \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n & = & b_1 \\
 & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n & = & b_2 \\
 & \ldots & & \\
 & a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n & = & b_m \\
 & x_j \geq 0 \text{ for } j = 1 \text{ to } n. & &
\end{array}
$$

DUAL PROBLEM:

$$
\begin{array}{llcl}
\textbf{min} & v = b_1y_1 + b_2y_2 + b_3y_3 + \ldots + b_my_m & & \\
\text{s.t.} & a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + \ldots + a_{m1}y_m & \geq & c_1 \\
 & a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + \ldots + a_{m2}y_m & \geq & c_2 \\
 & \ldots & & \\
 & a_{1n}y_1 + a_{2n}y_2 + a_{3n}y_3 + \ldots + a_{mn}y_m & \geq & c_n
\end{array}
$$

Weak Duality Theorem

Theorem. Suppose that $\bar{x}$ is any feasible solution to the primal, and $\bar{y}$ is any feasible solution to the dual. Then

$$\Sigma_{j=1..n} c_j \bar{x}_j \leq \Sigma_{i=1..n} \bar{y}_i b_i \qquad \text{(Max} \leq \text{Min)}$$

Proof.

$$\Sigma_{j=1..n} c_j \bar{x}_j \leq \Sigma_{j=1..n} \Sigma_{i=1..m} (\bar{y}_i a_{ij}) \bar{x}_j$$

$$\leq \Sigma_{i=1..m} \Sigma_{j=1..n} \bar{y}_i (a_{ij} \bar{x}_j)$$

$$\leq \Sigma_{i=1..m} \bar{y}_i b_i$$

Unboundedness Property

Theorem. Suppose that the primal (dual) problem has an unbounded solution. Then the dual (primal) problem has no feasible solution.

Proof.

Suppose that $\bar{y}$ was feasible for the dual. Then every solution to the primal problem is unbounded above by

$$\Sigma_{j=1..m} \bar{y}_i b_i .$$

Strong Duality Theorem

Theorem. If the primal problem has a finite optimal solution value, then so does the dual problem, and these two values are the same.

Strong Duality Illustrated

-z	x_1	x_2	x_3	x_4			Prices
1	0	-2/3	-1/3	0	=	-14/3	
0	1	1	1	1	=	1	-1/3
0	2	3	4	5	=	3	5/3

Optimal dual solution: $y_1 = -1/3$, $y_2 = 5/3$, $v = 14/3$

Optimal primal solution: $x_1 = 2/3$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1/3$ $z = 14/3$

Strong Duality Illustrated: the final tableau

-z	x_1	x_2	x_3	x_4		
1	0	-2/3	-1/3	0	=	-14/3
0	1	2/3	1/3	0	=	2/3
0	0	1/3	2/3	1	=	1/3

Observation

The cost coeffients satisfy complementary slackness

Optimal primal solution: $x_1 = 2/3$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1/3$ $v = 14/3$

Summary

- If we take an LP in standard form (max), we can formulate a dual problem
- ***Weak Duality:*** Each solution of the dual gives an upper bound on the maximum objective value for the primal
- ***Strong duality:*** if there are feasible solutions to the primal and dual, then the optimal objective value for both problems is the same.

Shadow prices solve the dual!

- **Theorem.** Suppose that the primal problem has a finite optimal solution value. Then the shadow prices for the primal problem form an optimal solution to the dual problem. (And the objective values are the same.)

David's Mineral Problem

-z	x_1	x_2	x_3	x_4			Shadow prices
1	3	4	6	8	=	0	
0	1	1	1	1	=	1	-1/3
0	2	3	4	5	=	3	5/3

-z	x_1	x_2	x_3	x_4		
1	0	-2/3	-1/3	0	=	-14/3
0	1	2/3	1/3	0	=	2/3
0	0	1/3	2/3	1	=	1/3

FACT: Optimal simplex multipliers are shadow prices.

Summary

- The dual problem to a maximizing LP provides upper bounds on the optimal objective function
- The maximum solution value for the primal is the same as the minimum solution value for the dual
- The shadow prices are optimal for the dual LP
- Next: alternative optimality conditions

Primal Problem

max $\Sigma_{j=1..m} c_j x_j$

s.t. $\Sigma_{j=1..n} a_{ij} x_j = b_i$ for i = 1 to m

$x_j \geq 0$ for all j = 1 to n

Dual Problem

min $\Sigma_{j=1..m} y_i b_i$

s.t. $\Sigma_{i=1..m} y_i a_{ij} \geq c_j$ for j = 1 to n

Optimality Condition 1.
A solution x* is optimal for the primal problem if it is a basic feasible solution and the tableau satisfies the optimality conditions.

Optimality Condition 2.
A solution x* is optimal for the primal problem if it is feasible and if there is a feasible solution y* for the dual with $\Sigma_{j=1..m} c_j x^*_j = \Sigma_{j=1..m} y^*_i b_i$

Primal Problem

max $\Sigma_{j=1..m} c_j x_j$

s.t. $\Sigma_{j=1..n} a_{ij} x_j = b_i$ for i = 1 to m

$x_j \geq 0$ for all j = 1 to n

Dual Problem

min $\Sigma_{j=1..m} y_i b_i$

s.t. $\Sigma_{i=1..m} y_i a_{ij} \geq c_j$ for j = 1 to n

Complementary Slackness Conditions.

Suppose that $\bar{y}$ is feasible for the dual, and let

$\bar{c}_j = c_j - \Sigma_{i=1..m} \bar{y}_i a_{ij}$.

Suppose $\bar{x}$ is feasible for the primal.

Theorem (complementary slackness). $\bar{x}$ and $\bar{y}$ are optimal for the primal and dual if and only if

$\bar{c}_j \bar{x}_j = 0$ for all j.

Complementary Slackness Illustrated

-z	x_1	x_2	x_3	x_4			Prices
1	0	-2/3	-1/3	0	=	-14/3	
0	1	2/3	1/3	0	=	2/3	-1/3
0	0	1/3	2/3	1	=	1/3	5/3

$\bar{c}$ =	0	-2/3	-1/3	0	$z = \Sigma_{i=1..n} \bar{c}_j \bar{x}_j + 14/3$
$\bar{x}$ =	2/3	0	0	1/3	Opt is $z = 14/3$

Next Lecture on Duality

- Dual linear programs in general form
- Optimality Conditions in general form.
- Illustrating Duality with 2-person 0-sum game theory

Coverage for first midterm
(Closed book exam)

- Formulations
- 2D graphing and finding opt. solution
- Setting up an LP in standard form
- The simplex algorithm starting with a bfs
- Phase 1 of the simplex algorithm
- Interpreting sensitivity analysis, including shadow prices, and ranges, and reduced costs
- Pricing out
- Using tableaus to determine shadow prices, reduced costs, and ranges