15.053 Thursday, March 7

- **Duality 2**
  - The dual problem, in general
  - illustrating duality with 2-person 0-sum game theory

Handouts: Lecture Notes

PRIMAL PROBLEM:

maximize $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$x_1 + x_2 + x_3 + x_4 = 1$

$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$

$x_1, x_2, x_3, x_4 \geq 0$

DUAL PROBLEM:

minimize	$y_1 + 3y_2$
Subject to	$y_1 + 2y_2 \geq 3$
	$y_1 + 3y_2 \geq 4$
	$y_1 + 4y_2 \geq 6$
	$y_1 + 5y_2 \geq 8$

Observation 1.

The constraint matrix in the primal is the transpose of the constraint matrix in the dual.

Observation 2.

The RHS coefficients in the primal become the cost coefficients in the dual.

PRIMAL PROBLEM:

maximize $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$x_1 + x_2 + x_3 + x_4 = 1$

$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$

$x_1, x_2, x_3, x_4 \geq 0$

DUAL PROBLEM:

minimize	$y_1 + 3y_2$
Subject to	$y_1 + 2y_2 \geq 3$
	$y_1 + 3y_2 \geq 4$
	$y_1 + 4y_2 \geq 6$
	$y_1 + 5y_2 \geq 8$

Observation 3. The cost coefficients in the primal become the RHS coefficients in the dual.

Observation 4. The primal (in this case) is a max problem with equality constraints and non-negative variables

The dual (in this case) is a minimization problem with ≥ constraints and variables unconstrained in sign.

PRIMAL PROBLEM:

maximize $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$x_1 + x_2 + x_3 + x_4 = 1$

$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$

$x_1, x_2, x_3, x_4 \geq 0$

DUAL PROBLEM:

minimize	$y_1 + 3y_2$
Subject to	$y_1 + 2y_2 \geq 3$
	$y_1 + 3y_2 \geq 4$
	$y_1 + 4y_2 \geq 6$
	$y_1 + 5y_2 \geq 8$

Question.

How does the dual change if we have inequality constraints?

Recall that the optimal dual variables are the shadow prices for the primal

PRIMAL PROBLEM:

				price
maximize	$z = 3x_1 + 4x_2 + 6x_3 + 8x_4$			
subject to	$x_1 + x_2 + x_3 + x_4$	$\geq$	$1+\Delta$	y_1
	$2x_1 + 3x_2 + 4x_3 + 5x_4$	$\leq$	$3+\Delta$	y_2
	x_1, x_2, x_3, x_4	$\geq$	0	

Method. Recall that the optimal dual variables are shadow prices.

Suppose we replace the 1 by $1 + \Delta$. Can the optimal objective value go down? Can it go up?

Conclusion: $y_1 \leq 0$.

Suppose we replace the 3 by $3 + \Delta$. Can the optimal objective value go down? Can it go up?

Conclusion: $y_2 \geq 0$.

PRIMAL PROBLEM:

maximize $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$x_1 + x_2 + x_3 + x_4 = 1$

$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$

$x_1, x_2, x_3, x_4 \geq 0$

DUAL PROBLEM:

minimize	$y_1 + 3y_2$
subject to	$y_1 + 2y_2 \geq 3$
	$y_1 + 3y_2 \geq 4$
	$y_1 + 4y_2 \geq 6$
	$y_1 + 5y_2 \geq 8$

Now suppose that we permit variables in the primal problem to be either ≤ 0 or unconstrained in sign.

Recall: reduced costs are the shadow prices of the "≥ 0" and "≤ 0" constraints.

PRIMAL PROBLEM:

		price
maximize	$z = 3x_1 + 4x_2 - 6x_3 + 8x_4$	
subject to	$x_1 + x_2 - x_3 + x_4 = 1$	y_1
	$2x_1 + 3x_2 - 4x_3 + 5x_4 = 3$	y_2
	$x_1 \geq 0$ $x_2 \geq 0$ $x_3 \leq 0$ x_4 uis	

$\bar{c}_1 = 3 - y_1 - 2y_2$

Suppose we replace "$x_1 \geq 0$" by "$x_1 \geq \Delta$". Can the optimal objective value go down? Can it go up?

Conclusion: $\bar{c}_1 \leq 0$, and thus $y_1 + 2y_2 \geq 3$.

To do with your partner: figure out the sign on the shadow price for the constraint $x_3 \leq 0$. Also, what do we do with x_4 uis?

Summary for forming the dual of a maximization problem

PRIMAL		DUAL
Max		Min
$\Sigma_j a_{ij}x_j = b_i$	➔	y_i u.i.s.
$\Sigma_j a_{ij}x_j \geq b_i$	➔	$y_i \leq 0$
$\Sigma_j a_{ij}x_j \leq b_i$	➔	$y_i \geq 0$
$x_j \geq 0$	➔	$\Sigma_j y_i a_{ij} \geq c_j$
$x_j \leq 0$	➔	$\Sigma_j y_i a_{ij} \leq c_j$
x_j u.i.s.	➔	$\Sigma_j y_i a_{ij} = c_j$

Complementary Slackness

PRIMAL	DUAL	Comp. Slackness
$\Sigma_j a_{ij}x_j \geq b_i$	$y_i \leq 0$	$y_i (\Sigma_j a_{ij}x_j - b_i) = 0$
$\Sigma_j a_{ij}x_j \leq b_i$	$y_i \geq 0$	$y_i (\Sigma_j a_{ij}x_j - b_i) = 0$
$x_j \geq 0$	$\Sigma_j y_i a_{ij} \geq c_j$	$x_j (\Sigma_j y_i a_{ij} - c_j) = 0$
$x_j \leq 0$	$\Sigma_j y_i a_{ij} \leq c_j$	$x_j (\Sigma_j y_i a_{ij} - c_j) = 0$

Determine the dual

PRIMAL PROBLEM:

maximize	$z = 3x_1 + c\, x_2 + 6x_3$
subject to	$x_1 + 2x_2 + x_3 \geq 1$
	$2x_1 + 3x_2 + 4x_3 = 3$
	$x_1 \geq 0$, $x_2 \leq 0$, x_3 u.i.s.

Determine the dual of the above linear program. Then compare your result with that of your partner.

Duals of Minimization Problem

- **The dual of a minimization problem is a maximization problem.**
- **The shadow prices for the dual linear program form the optimal solution for the primal problem.**
- **The dual of the dual is the primal.**

2-person 0-sum game theory

Person R chooses a row: either 1, 2, or 3

Person C chooses a column: either 1, 2, or 3

-2	1	2
2	-1	0
1	0	-2

This matrix is the payoff matrix for player R. (And player C gets the negative.)

e.g., R chooses row 3; C chooses column 1

R gets 1; C gets –1 (zero sum)

Some more examples of payoffs

R chooses 2, C chooses 3

R gets 0; C gets 0 (zero sum)

-2	1	2
2	-1	0
1	0	-2

R chooses row 3; C chooses column 3

R gets -2; C gets +2 (zero sum)

Next: 2 volunteers

Player R puts out 1, 2 or 3 fingers

Player C simultaneously puts out 1, 2, or 3 fingers

-2	1	2
2	-1	0
1	0	-2

We will run the game for 5 trials.

R tries to maximize his or her total

C tries to minimize R's total.

Next: Play the game with your partner (If you don't have one, then watch)

Player R puts out 1, 2 or 3 fingers

Player C simultaneously puts out 1, 2, or 3 fingers

-2	1	2
2	-1	0
1	0	-2

We will run the game for 5 trials.

R tries to maximize his or her total

C tries to minimize R's total.

Who has the advantage: R or C?

Suppose that R and C are both brilliant players and they play a VERY LONG TIME.

-2	1	2
2	-1	0
1	0	-2

We will find a lower and upper bound on the payoff to R using linear programming.

Will R's payoff be positive in the long run, or will it be negative, or will it converge to 0?

Computing a lower bound

Suppose that player R must announce his or her strategy in advance of C making a choice.

-2	1	2
2	-1	0
1	0	-2

If R is forced to announce a row, then what row will R select?

A strategy that consists of selecting the same row over and over again is a "pure strategy." R can guarantee a payoff of at least –1.

Computing a lower bound on R's payoff

Suppose we permit R to choose a random strategy.

-2	1	2
2	-1	0
1	0	-2

Suppose R will flip a coin, and choose row 1 if Heads, and choose row 3 if tails.

The column player makes the choice after hearing the strategy, but before seeing the flip of the coin.

What is player's C best response?

				Prob.
	-2	1	2	.5
	2	-1	0	0
	1	0	-2	.5
Expected Payoff	-.5	.5	0	

If C knows R's random strategy, then C can determine the expected payoff for each column chosen.

What would C's best response be?

So, with a random strategy R can get at least -.5

Suppose that R randomizes between row 1 and row 2.

				Prob.
	-2	1	2	.5
	2	-1	0	.5
	1	0	-2	0
Expected Payoff	0	0	1	

What would C's best response be?

So, with a random strategy R can get at least 0.

What is R's best random strategy?

				Prob.
	-2	1	2	x_1
	2	-1	0	x_2
	1	0	-2	x_3
Expected Payoff	A	B	C	

$x_1 + x_2 + x_3 = 1$

C will choose the column that is the minimum of A, B, and C.

A: $-2x_1 + 2x_2 + x_3$

B: $x_1 - x_2$

C: $2x_1 - 2x_3$

Finding the min of 2 numbers as an optimization problem.

Let z = min (x, y)

Then z is the optimum solution value to the following LP:

maximize z

subject to $z \le x$

$z \le y$

R's best strategy, as an LP

	-2	1	2	x_1
	2	-1	0	x_2
	1	0	-2	x_3
Expected Payoff	A	B	C	

Maximize	z (the payoff to x)
A:	$z \le -2x_1 + 2x_2 + x_3$
B:	$z \le x_1 - x_2$
C:	$z \le 2x_1 - 2x_3$
	$x_1 + x_2 + x_3 = 1$
	$x_1, x_2, x_3 \ge 0$

2-person 0-sum game

The Row Player's LP, in general

	a_{11}	a_{12}	a_{13}	…	a_{1m}	x_1
	a_{21}	a_{22}	a_{23}	…	a_{2m}	x_2
		…				
	a_{n1}	a_{n2}	a_{n3}	…	a_{nm}	x_n
Expected Payoff	P_1	P_2	P_3	…	P_m	

Maximize	z (the payoff to x)
P_j:	$z \le a_{1j} x_1 + a_{2j} x_2 + \ldots + a_{nj} x_n$ for all j
	$x_1 + x_2 + \ldots + x_n = 1$
	$x_j \ge 0$ for all j

Here is the ***optimal*** random strategy for R.

				Prob.
	-2	1	2	7/18
	2	-1	0	5/18
	1	0	-2	1/3
Expected Payoff	1/9	1/9	1/9	

This strategy is a lower bound on what R can obtain if he or she takes into account what C is doing.

The optimal payoff to R is 1/9.

So, with a random strategy R guarantees obtaining at least 1/9.

We can obtain a lower bound for C (and an upper bound for R) in the same manner.

				Exp. payoff
	-2	1	2	1/3
	2	-1	0	1/3
	1	0	-2	-1/3
Prob.	1/3	1/3	1/3	

If C chooses a random strategy, it will give an upper bound on what R can obtain.

If C announced this random strategy, R would select 1 or 2.

So, with this random strategy C guarantees that R obtains at most 1/3.

The best random strategy for C is to minimize the max expected payoff.

				Exp. payoff
	-2	1	2	1/9
	2	-1	0	1/9
	1	0	-2	1/9
Prob.	1/3	5/9	1/9	

The column player can set up a linear program to determine the best random strategy

Note: C can use a random strategy to ensure that R receives at most 1/9 on average.

So, with this random strategy R gets only 1/9.

The best random strategy for C is to minimize the max expected payoff.

				Exp. payoff
	-2	1	2	1/9
	2	-1	0	1/9
	1	0	-2	1/9
Prob.	1/3	5/9	1/9	

So, the optimal average payoff to the game is 1/9, assuming that both players play optimally.

For 2-person 0-sum games, the maximum payoff that R can guarantee by choosing a random strategy is the minimum payoff to R that C can guarantee by choosing a random strategy.

2-person 0-sum games in general.

- Let x denote a random strategy for R, with value z(x) and let y denote a random strategy for C with value v(y).
- z(x) ≤ v(y) for all x, y
- The optimum x* can be obtained by solving an LP. So can the optimum y* .
- z(x*) = v(y*)
- The two linear programs are dual to each other.

More on 2-person 0-sum games

- In principle, R can do as well with a random fixed strategy as by carefully varying a strategy over time
- Duality for game theory was discovered by Von Neumann and Morgenstern (predates LP duality).
- The idea of randomizing strategy permeates strategic gaming.