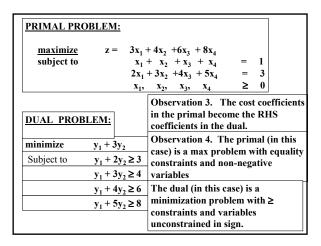
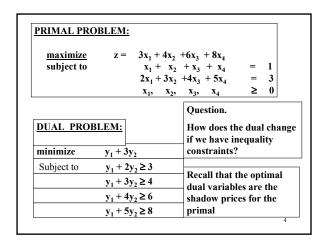
15.053 Thursday, March 7

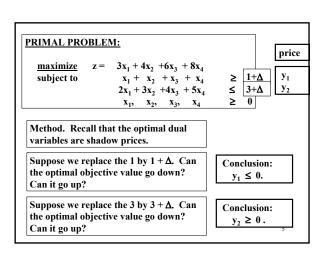
- Duality 2
 - The dual problem, in general
 - illustrating duality with 2-person 0-sum game theory

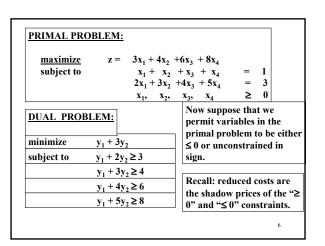
Handouts: Lecture Notes

PRIMAL PRO	$\mathbf{z} = 3\mathbf{x}_1 + 4\mathbf{x}_2$. +6x. +8x.	
subject to	•	$\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} = \frac{1}{1}$	
subject to		$\frac{1}{2} + 4x_3 + 5x_4 = 3$	
	•	$x_{3}, x_{4} \ge 0$	
		Observation 1.	
DUAL PROBLEM:		The constraint matrix in the primal is the transpose of	
minimize	$y_1 + 3y_2$	the constraint matrix in the	
Subject to	$y_1 + 2y_2 \ge 3$	dual.	
	$y_1 + 3y_2 \ge 4$	Observation 2.	
	$y_1 + 4y_2 \ge 6$	The RHS coefficients in the	
	$y_1 + 5y_2 \ge 8$	primal become the cost coefficients in the dual.	
		coefficients in the dual.	









PRIMAL PROBLEM:

 $\begin{array}{llll} \underline{\text{maximize}} & z = & 3x_1 + 4x_2 - 6x_3 + 8x_4 \\ \text{subject to} & & x_1 + x_2 - x_3 + x_4 = 1 \\ & & 2x_1 + 3x_2 - 4x_3 + 5x_4 = 3 \\ & & x_1 \ge 0 \ x_2 \ge 0 \ x_3 \le 0 \ x_4 \ \text{uis} \end{array}$

 $\overline{c_1} = 3 - y_1 - 2y_2$

Suppose we replace " $x_1 \ge 0$ " by " $x_1 \ge \Delta$ ". Can the optimal objective value go down? Can it go up?

Conclusion: $\overline{c_1} \le 0$, and thus $y_1 + 2y_2 \ge 3$.

price

 \mathbf{y}_1

To do with your partner: figure out the sign on the shadow price for the constraint $x_3 \le 0$. Also, what do we do with x_4 uis?

Summary for forming the dual of a maximization problem

$\begin{array}{c} \text{PRIMAL} \\ \text{Max} \\ \Sigma_j \, a_{ij} x_j \, = \, b_i \\ \Sigma_j \, a_{ij} x_j \, \ge \, b_i \\ \end{array} \quad \begin{array}{c} \rightarrow \\ \Sigma_j \, a_{ij} x_j \, \le \, b_i \\ x_j \, \ge \, 0 \\ x_j \, \le \, 0 \\ x_j \, \subseteq \, 0 \end{array} \quad \begin{array}{c} \rightarrow \\ \rightarrow \\ x_j \, \subseteq \, 0 \\ x_j \, \cup \, 1.5. \end{array}$

 $\begin{aligned} & \text{Min} \\ & y_i \text{ u.i.s.} \\ & y_i \leq 0 \\ & y_i \geq 0 \\ & \Sigma_j \text{ y}_i a_{ij} \geq c_j \\ & \Sigma_j \text{ y}_i a_{ij} \leq c_j \\ & \Sigma_j \text{ y}_i a_{ij} = c_j \end{aligned}$

Complementary Slackness

$\label{eq:continuity} \begin{array}{lll} \mbox{PRIMAL} & \mbox{DUAL} \\ \Sigma_j \, a_{ij} x_j \, \geq \, b_i & y_i \leq \, 0 \\ \\ \Sigma_j \, a_{ij} x_j \, \leq \, b_i & y_i \geq \, 0 \\ \\ x_j \, \geq \, 0 & \Sigma_j \, y_i a_{ij} \, \geq \, c_j \\ \\ x_j \, \leq \, 0 & \Sigma_j \, y_i a_{ij} \, \leq \, c_j \end{array}$

Comp. Slackness
$$y_{i} (\Sigma_{j} a_{ij}x_{j} - b_{i}) = 0$$

$$y_{i} (\Sigma_{j} a_{ij}x_{j} - b_{i}) = 0$$

$$x_{j} (\Sigma_{j} y_{i}a_{ij} - c_{j}) = 0$$

$$x_{j} (\Sigma_{j} y_{i}a_{ij} - c_{j}) = 0$$

Determine the dual

PRIMAL PROBLEM:

 $\begin{array}{lll} \underline{\text{maximize}} & z = & 3x_1 + c \; x_2 \; + 6x_3 \\ \text{subject to} & & x_1 + \; 2x_2 \; + x_3 & \geq \; 1 \\ & & 2x_1 + \; 3x_2 \; + 4x_3 & = \; 3 \\ & & x_1 \geq \; 0, \; \; x_2 \leq \; 0, \quad \; x_3 \; \text{u.i.s.}. \end{array}$

Determine the dual of the above linear program. Then compare your result with that of your partner.

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Duals of Minimization Problem

- The dual of a minimization problem is a maximization problem.
- The shadow prices for the dual linear program form the optimal solution for the primal problem.
- The dual of the dual is the primal.

2-person 0-sum game theory

Person R chooses a row: either 1, 2, or 3

Person C chooses a column: either 1, 2, or 3

 -2
 1
 2

 2
 -1
 0

 1
 0
 -2

This matrix is the payoff matrix for player R. (And player C gets the negative.)

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e.g., R chooses row 3; C chooses column 1

R gets 1; C gets -1 (zero sum)

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Some more examples of payoffs

R chooses 2, C chooses 3

R gets 0; C gets 0 (zero sum)

-2	1	2
2	-1	0
1	0	-2

R chooses row 3; C chooses column 3

R gets -2; C gets +2 (zero sum)

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Next: 2 volunteers

Player R puts out 1, 2 or 3 fingers

Player C simultaneously puts out 1, 2, or 3 fingers

-2	1	2
2	-1	0
1	0	-2

We will run the game for 5 trials.

R tries to maximize his or her total

C tries to minimize R's total.

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Next: Play the game with your partner (If you don't have one, then watch)

Player R puts out 1, 2 or 3 fingers

Player C simultaneously puts out 1, 2, or 3 fingers

-2	1	2
2	-1	0
1	0	-2

We will run the game for 5 trials.

R tries to maximize his or her total

C tries to minimize R's total.

Who has the advantage: R or C?

Suppose that R and C are both brilliant players and they play a VERY LONG TIME.

-2	1	2
2	-1	0
1	0	-2

We will find a lower and upper bound on the payoff to R using linear programming.

Will R's payoff be positive in the long run, or will it be negative, or will it converge to 0?

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Computing a lower bound

Suppose that player R must announce his or her strategy in advance of C making a choice.

-2	1	2
2	-1	0
1	0	-2

If R is forced to announce a row, then what row will R select?

A strategy that consists of selecting the same row over and over again is a "pure strategy." R can guarantee a payoff of at least -1.

Computing a lower bound on R's payoff

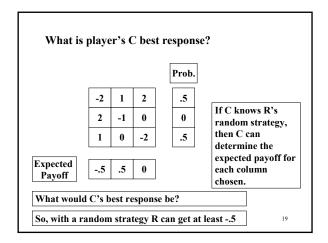
Suppose we permit R to choose a random strategy.

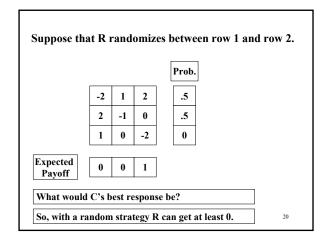
-2	1	2
2	-1	0
1	0	-2

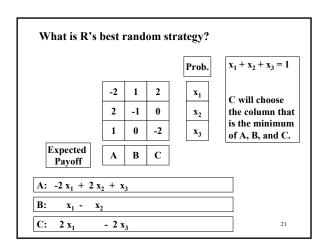
Suppose R will flip a coin, and choose row 1 if Heads, and choose row 3 if tails.

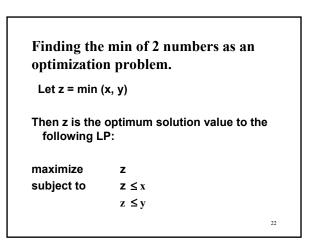
The column player makes the choice after hearing the strategy, but before seeing the flip of the coin.

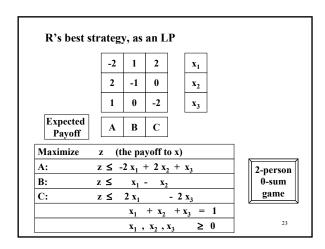
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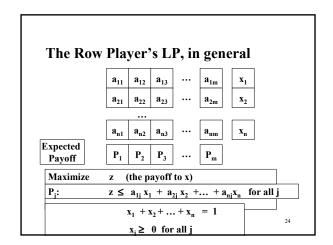


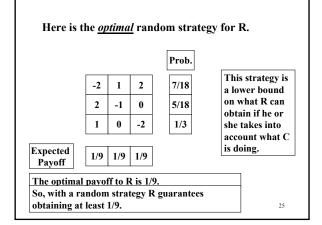


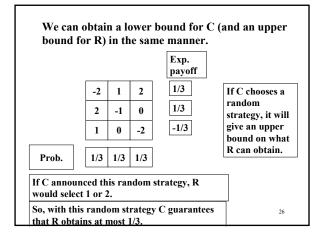


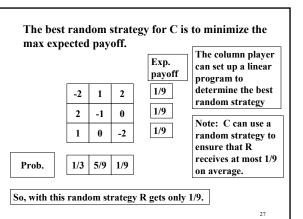


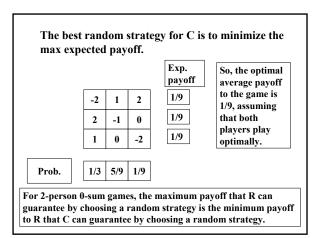












2-person 0-sum games in general.

- Let x denote a random strategy for R, with value z(x) and let y denote a random strategy for C with value v(y).
- $z(x) \le v(y)$ for all x, y
- The optimum x* can be obtained by solving an LP. So can the optimum y*.
- \bullet z(x*) = v(y*)
- The two linear programs are dual to each other.

More on 2-person 0-sum games

- In principle, R can do as well with a random fixed strategy as by carefully varying a strategy over time
- Duality for game theory was discovered by Von Neumann and Morgenstern (predates LP duality).
- The idea of randomizing strategy permeates strategic gaming.

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