15.053 Thursday, April 11

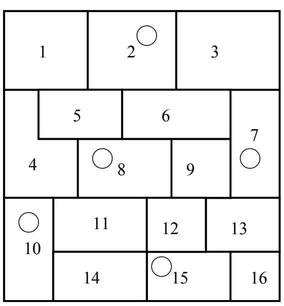
- Some more applications of integer programming
- Cutting plane techniques for getting improved bounds

Handouts: Lecture Notes

Example: Fire company location.

- Consider locating fire companies in different districts.
- Objective: place fire companies so that each district either has a fire company in it, or one that is adjacent, and so as to minimize cost.

Example for the Fire Station Problem



Let $x_j = 1$ if a fire station is placed in district j. $x_j = 0$ otherwise.

Let c_j = cost of putting a fire station is district j.

With partners, formulate the fire station problem.

Set covering problem

- set S= {1, …, m} of items to be covered
  - districts that need to have a fire station or be next to a district with one
- n subsets of S
  - For each possible location j for a fire station, the subset is district j plus the list of districts that are adjacent to district j.
  - a_{ij} = 1 if district i is adjacent to district j or if i = j.

Minimize $\sum_j c_j x_j$

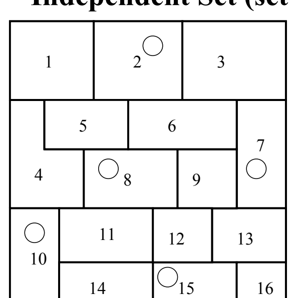
subject to $\sum_j a_{ij} x_j \geq 1$ for each i

x_j is binary for each j

Covering constraints are common

- Fleet routing for airlines:
  - Assigning airplanes to flight legs.
  - Each flight must be included
- Assigning crews to airplanes
  - Each plane must be assigned a crew
- Warehouse location
  - Each retailer needs to be served by some warehouse

Independent Set (set packing)



What is the maximum number of districts no two of which have a common border?

With your partner, formulate the independent set problem. List all constraints containing x_{10}.

Set Packing constraints arise often in manufacturing and logistics

- Cutting shapes out of sheet metal
- Manufacturing many items at once which share resources (how much work can be done in parallel?)

Review from Last Lecture

Investment budget = $14,000

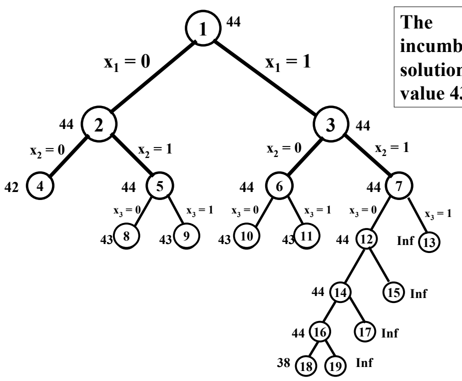
Investment	1	2	3	4	5	6
Cash Required (1000s)	$5	$7	$4	$3	$4	$6
NPV added (1000s)	$16	$22	$12	$8	$11	$19

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \le 14$

x_j **binary for j = 1 to 6**

Branch and Bound



The incumbent solution has value 43

On Bounds from Linear Programming

We found an incumbent with a value of 43.

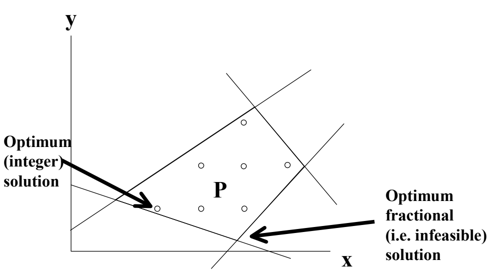
The best LP solution value is between 44 and 45.

Is there a way that we could have directly established an upper bound between 43 and 44? Perhaps we could form a "better linear program."

The closer the LP is to the Integer Program, the better.

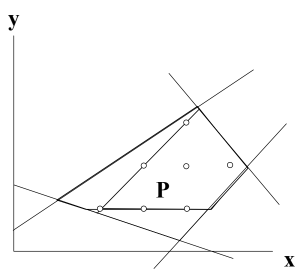
Note: not all LP formulations are the same in terms of "closeness"

Using Cutting Planes



Example. Minimize $x + 10y$
subject to x, y **are in P**
x, y **integer**

Using Cutting Planes



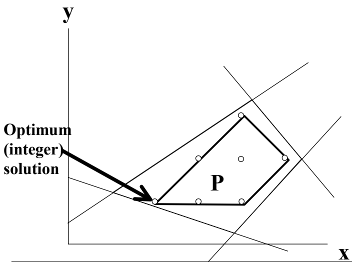
Idea: add constraints that eliminate fractional solutions to the LP without eliminating any integer solutions.

add "$y \ge 1$"

add "$y \le x - 1$"

Example. Minimize $x + 10y$
subject to x, y **are in P**
x, y **integer**

Using Cutting Planes



If we add exactly the right inequalities, then every corner point of the LP will be integer, and the IP can be solved by solving the LP

We call this minimal LP, the *convex hull* of the IP solutions.

Example. Minimize $x + 10y$
subject to x, y are in P
x, y integer

For large problems, these constraints are hard to find.

More on adding constraints

- **The tightest possible constraints are very useful, and are called <u>facets</u>.**
- **Suppose that we are maximizing, and z_{LP} is the opt for the LP relaxation, and z_{IP} is the opt for the IP. Then $z_{IP} \leq z_{LP}$**
- **Ideally, we want z_{IP} to be close to z_{LP}. This is GREAT for branch and bound.**
- **Adding lots of valid inequalities can be very helpful.**
  - It has no effect on z_{IP}.
  - It can reduce z_{LP} significantly.

"Pure" Cutting Plane Technique

- **Instead of partitioning the feasible region, the (pure) cutting plane technique works with a single LP**
- **It adds cutting planes (valid linear programming inequalities) to this LP iteratively.**
- **At each iteration the feasible region is successively reduced until an integer optimal is found by solving the LP.**
- **In practice, it is also used as part of branch and bound. The essential idea is finding valid "cuts" or inequalities.**

Where do these cuts come from?

- **Two approaches**
  - Problem specific
    - Illustrated on Traveling Salesman Problem and Knapsack Problem
  - LP-based approach, that works for general integer programs
    - Gomory cutting planes

Problem Specific

- **The capital budgeting (knapsack) problem**

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

x_j binary for j = 1 to 6

The LP Relaxation

- **The capital budgeting (knapsack) problem**

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$0 \leq x_j \leq 1$ for j = 1 to 6

The optimal solution: $x_1 = 1, x_2 = 3/7, x_3 = 0$
$x_4 = 0, x_5 = 0, \quad x_6 = 1$

Can we find a valid inequality (cut) that eliminates this solution?

The LP Relaxation

- The capital budgeting (knapsack) problem

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$0 \leq x_j \leq 1$ for j = 1 to 6

The optimal solution: $x_1 = 1, x_2 = 3/7, x_3 = 0$
$x_4 = 0, x_5 = 0, x_6 = 1$ $z = 44\ 3/7$

$5x_1 + 7x_2 + 6x_6 \leq 14 \;\Rightarrow\; x_1 + x_2 + x_6 \leq 2$

Excel

After one cut

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$x_1 + x_2 + x_6 \leq 2$

$0 \leq x_j \leq 1$ for j = 1 to 6

The optimal solution: $x_1 = 0, x_2 = 1, x_3 = 1/4$
$x_4 = 0, x_5 = 0, x_6 = 1$ $z = 44$

$7x_2 + 4x_3 + 6x_6 \leq 14 \;\Rightarrow$??

Excel

After two cuts

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$x_1 + x_2 + x_6 \leq 2$

$x_2 + x_3 + x_6 \leq 2$

$0 \leq x_j \leq 1$ for j = 1 to 6

The optimal solution: $x_1 = 1/3, x_2 = 1, x_3 = 1/3$
$x_4 = 0, x_5 = 0, x_6 = 1$ $z = 44$

$5x_1 + 7x_2 + 4x_3 + 6x_6 \leq 14 \;\Rightarrow$??

Excel

Obtaining the valid cut

It is easy to maximize

$x_1 + x_2 + x_3 + x_6$

s.t. $5x_1 + 7x_2 + 4x_3 + 6x_6 \leq 14$

x_j binary for j = 1, 2, 3, 6.

Greedily put the smallest budget items in the knapsack until no more can be put in.

In this case, items 1 and 3 are put in, and there is no room for item 2 or 6.

So $x_1 + x_2 + x_3 + x_6 \leq 2$.

After "three" cuts

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$x_1 + x_2 + x_6 \leq 2$

$x_2 + x_3 + x_6 \leq 2$

$x_1 + x_2 + x_3 + x_6 \leq 2$

$0 \leq x_j \leq 1$ for j = 1 to 6

Note: the new cuts dominates the other cuts.

Excel

Eliminating the redundant constraints

maximize $16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14$

$x_1 + x_2 + x_3 + x_6 \leq 2$

$0 \leq x_j \leq 1$ for j = 1 to 6

The optimal solution: $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0,$
$x_5 = 1/4, x_6 = 1$ $z = 43\ 3/4$

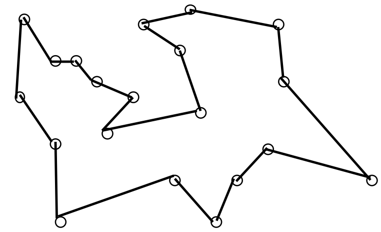
$\boxed{\text{So, } z^* \leq 43}$

Excel

Summary for knapsack problem

- We could find some simple valid inequalities that showed that $z^* \leq 43$. This is the optimal objective value.
- It took 3 cuts
  - Had we been smarter it would have taken 1 cut
- We had a simple approach for finding cuts.
  - This does not find all of the cuts.
- Recall, it took 25 nodes of a branch and bound tree
- In fact, researchers have found cutting plane techniques to be necessary to solve large integer programs (usually as a way of getting better bounds.)

Traveling Salesman Problem (TSP)



What is a minimum length tour that visits each point?

Comments on the TSP

- Very well studied problem
- It's often the problem for testing out new algorithmic ideas
- NP-complete (it is intrinsically difficult in some technical sense)
- Large instances have been solved optimally (5000 cities and larger)
- Very large instances have been solved approximately (10 million cities to within a couple of percent of optimum.)
- We will formulate it by adding constraints that look like cuts

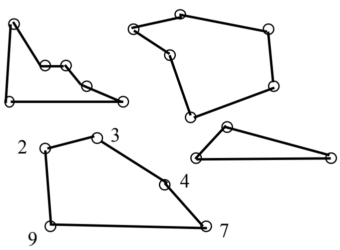
The TSP as an IP, almost

- Let $x_e = 1$ if arc e is in the tour
  $x_e = 0$ otherwise
- Let A(i) = arcs incident to node i
- Minimize $\Sigma_e \; c_e \, x_e$
- subject to $\Sigma_{e \in A(i)} \; x_e = 2$

  x_e is binary

Are these constraints enough?

Subtours

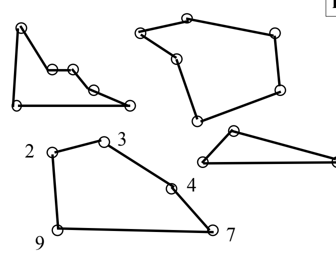


Any integer solution with exactly two points incident to every node is the union of tours. Why?

Objective: add constraints that eliminate these "subtours" but that do not eliminate any TSP tour.

Subtours

Let S = {2, 3, 4, 7, 9}. Then any tour has at most 4 arcs in S.



For each strict subset S of cities, add the constraint that the number of arcs in S is at most |S| - 1.

This ensures that the set S will not have a subtour going through all five nodes.

Formulation of the TSP as an IP

- Minimize $\sum_e c_e x_e$
- subject to $\sum_{e \text{ inc to } i} x_e = 2$
- $\sum_{e \text{ in } S} x_e \leq |S| - 1$

  subtour breaking constraints

  x_e is binary

Are these constraints enough?

Yes! Unfortunately, there are exponentially many.

But it works well anyway. Generate as needed.

More on the TSP

- The subtour elimination constraints are good in the sense that for practical problem the LP bound is usually 1% to 2% from the optimal TSP tour length.
- One can add even more complex constraints, and people do. (and it helps)

Gomory Cuts: a method of generating cuts using LP tableaus

- Consider the following constraint in an integer program

x_1	x_2	x_3	x_4		
1 3/5	4 1/5	3	2/5	=	9 4/5

Focus on the fractions.

➔ $3/5\ x_1 + 1/5\ x_2 + 2/5\ x_4 =$ Integer $+ 4/5$

➔ $3/5\ x_1 + 1/5\ x_2 + 2/5\ x_4 \geq 4/5$

➔ $3\ x_1 + x_2 + 2\ x_4 \geq 4.$

Gomory Cuts

What did we rely on to obtain the Gomory cut?

x_1	x_2	x_3	x_4		
1 3/5	4 1/5	3	2/5	=	9 4/5

A single constraint with a fractional right hand side

All coefficients of the constraint were positive.

All variables must be integral

What to do if coefficients are negative

x_1	x_2	x_3	x_4		
1 3/5	-4 3/5	-3	-2/5	=	-1 1/5

Rewrite so that the coefficients of the fractional parts are positive.

1 3/5	-5 +2/5	-3	-1 +3/5	=	-2 +4/5

➔ $3/5\ x_1 + 2/5\ x_2 + 3/5\ x_4 =$ integer $+ 4/5$.

➔ $3/5\ x_1 + 2/5\ x_2 + 3/5\ x_4 \geq 4/5$.

➔ $3\ x_1 + 2\ x_2 + 3\ x_4 \geq 4$.

In general

x_1	x_2	x_3	x_4		
1 3/5	-4 3/5	-3	-2/5	=	-1 1/5
3/5	2/5	0	+3/5	$\geq$	+4/5

x_1	x_2	x_3	x_4		
$\bar{a}_{i1}$	$\bar{a}_{i2}$	$\bar{a}_{i3}$	$\bar{a}_{i4}$	=	$\bar{b}_i$
fr($\bar{a}_{i1}$)	fr($\bar{a}_{i2}$)	fr($\bar{a}_{i3}$)	fr($\bar{a}_{i4}$)	$\geq$	fr($\bar{b}_i$)

Let fr(a) be the positive fractional part of a ;

fr(a) = a - $\lfloor a \rfloor$

fr(2 3/5) = 2 3/5 – 2 = 3/5

fr(-2 3/5) = - 2 3/5 – (-3) = 2/5

How to generate cutting planes?

- In general
  - After pivoting, find a basic variable that is fractional.
  - Write a Gomory cut. (It is an inequality constraint, leading to a new slack variable).
  - Note: the only coefficients that are fractional correspond to non-basic variables (why?)
  - The Gomory cut makes the previous basic feasible solution infeasible. (why?)
  - Resolve the LP with the new constraint, and iterate.

Integer Programming Summary

- Dramatically improves the modeling capability
  - Economic indivisibilities
  - Logical constraints
  - Modeling non-linearities
- Not as easy to model.
- Not as easy to solve.

IP Solution Techniques Summary

- Branch and Bound
  - very general and adaptive
  - used in practice (e.g. Excel Solver)
- Implicit Enumeration
  - branch and bound technique for binary IPs
- Cutting planes
  - clever way of improving bounding
  - active area for research, theoretical and applied