#### 15.053

### **February 7, 2002**

- A brief review of Linear Algebra
- Linear Programming Models

**Handouts: Lecture Notes** 

### Review of Linear Algebra

- Some elementary facts about vectors and matrices.
- The Gauss-Jordan method for solving systems of equations.
- Bases and basic solutions and pivoting.

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### **Elementary Facts about Vectors**

 $v = [v_1 \quad v_2 \quad v_3 \quad v_4]$  is called a <u>row vector</u>.

The <u>transpose</u> of v is a <u>column vector</u>.  $v' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ 

 $w = [w_1 \ w_2 \ w_3 \ w_4]$  is another row vector.

The <u>inner product</u> of vectors w and v is given by:  $v \circ w = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$ 

### **Matrix Multiplication**

$$A=(a_{ij})$$
  $B=(b_{ij})$   $C=(c_{ij})=A\times B$ 

Suppose that A has n columns and B has n rows.

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

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### **Multiplying Matrices**

Let  $C = (c_{ij}) = AB$ . Then  $c_{ij}$  is the *inner* product of row i of A and column j of B.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \boldsymbol{a}_{13} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \boldsymbol{a}_{23} \\ \boldsymbol{a}_{31} & \boldsymbol{a}_{32} & \boldsymbol{a}_{33} \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_{11} & \boldsymbol{b}_{12} & \boldsymbol{b}_{13} \\ \boldsymbol{b}_{21} & \boldsymbol{b}_{22} & \boldsymbol{b}_{23} \\ \boldsymbol{b}_{31} & \boldsymbol{b}_{32} & \boldsymbol{b}_{33} \end{bmatrix}$$

For example, what is  $c_{23}$ ?

$$c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$$

### **Multiplying Matrices**

Let C =  $(c_{ij})$  = A × B. Then each column of C is obtained by adding multiples of columns of A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \times \begin{bmatrix} 100 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 123 \\ 456 \\ 789 \end{bmatrix}$$
$$= 100 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 10 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Similarly, each row of C is obtained by adding multiples of rows of B.

## **Elementary Facts about Solving Equations**

Solve for 
$$Ax = b$$
, where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$2 \times 3$$

$$3 \times 1$$

$$x_1 + 2x_2 + 4x_3 = 0$$
  
 $2x_1 + x_2 - x_3 = 6$ 

Find a linear combination of the columns of *A* that equals *b*.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

### **Solving a System of Equations**

	x1	x2	х3	х4				
	$\bigcap$ 1	2	4	1	=		0	
Γ	2	1	-1	-1	=		6	Г
Г	-1	1	2	2	=		-3	Г
Г	$\overline{}$			7		Г		

To solve a system of equations, use Gauss-Jordan elimination.

### The system of equations

<b>x</b> <sub>1</sub>	X <sub>2</sub>	<b>x</b> <sub>3</sub>	<b>x</b> <sub>4</sub>			
				_		
1	2	4	1		=	0
2	1	-1	-1		=	6
-1	1	2	2		=	-3
					ı	

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### Pivot on the element in row 1 column 1

Subtract 2 times constraint 1 from constraint 2. Add constraint 1 to constraint 3.

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### Pivot on the element in Row 2, Column 2

Divide constraint 2 by -3. Subtract multiples of constraint 2 from constraints 1 and 3.

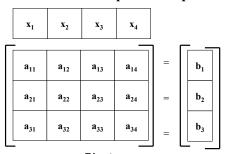
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### Pivot on the element in Row 3, Column 3

Divide constraint 3 by -3. Add multiples of constraint 3 to constraints 1 and 2.

What is a solution to this system of equations?

### The fundamental operation: pivoting



Pivot on a<sub>23</sub>

### Pivot on a23

 $\overline{a}_{11} = a_{11} - a_{13}(a_{21}/a_{23})$ 

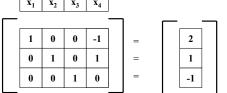
 $b_2/a_{23}$ 

- a <sub>11</sub>	a <sub>12</sub>	0	a <sub>14</sub>	=
a <sub>21</sub> /a <sub>23</sub>	a <sub>22</sub> /a <sub>23</sub>	1	a <sub>24</sub> /a <sub>23</sub>	=
a <sub>31</sub>	a <sub>32</sub>	0	a <sub>34</sub>	=

What will be the next coefficient of  $b_1$ ?  $a_{32}$ ? of  $a_{ij}$  for  $i \neq 2$ ?

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### Jordan Canonical Form for an m x n matrix



There are m columns that have been transformed into unit vectors, one for each row. The variables in these columns are called "basic."

The "basic" solution is  $x_1 = 2$ ,  $x_2 = 1$ ,  $x_3 = -1$   $x_4 = 0$  15

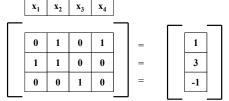
### There is an easily determined solution for every choice of non-basic variables.

The remaining variable x<sub>4</sub> is called non-basic.

If we set  $x_4 = 2$ , what solution do we get?

If we set  $x_4 = \Delta$ , what solution do we get?

# **Another Jordan Canonical Form for the same system of equations**



What are the "basic" variables?

What is the basic solution?

### **Applications**

- A Financial Model
- Scheduling Postal Workers

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### A Financial Problem

- Sarah has \$1.1 million to invest in five different projects for her firm.
- Goal: maximize the amount of money that is available at the beginning of 2005.
  - (Returns on investments are on the next slide).
- At most \$500,000 in any investment
- Can invest in CDs, at 5% per year.

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### Return on investments (undiscounted dollars)

	Α	В	С	D	E
Jan. 2002	-1	-	-1	-1	-
Jan. 2003	.4	-1	1.2	-	-
Jan. 2004	.8	.4	-	-	-1
Jan. 2005	•	.8	-	1.5	1.2

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### Formulate Sarah's problem as an LP

- Payback for A: for every dollar invested in January of 2002, Sarah receives \$.40 in January of 2003 and \$.80 in January of 2004.
- FORMULATION.
  - STEP 1. Choose the decision variables
  - Let x<sub>A</sub> denote the amount in millions of dollars invested in A.
  - Define  $x_B$ ,  $x_C$ ,  $x_D$ , and  $x_E$  similarly.
  - Let  $\mathbf{x}_2$  denote the amount put in a CD in 2002. (Define  $\mathbf{x}_3$  and  $\mathbf{x}_4$  similarly)

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### Formulating the model

- With your partner formulate the LP model.
- Step 2. Formulate the objective function
  - put the objective function in words first. E.G. we are "minimizing cost" or "maximizing utility"
- Step 3. Formulate the constraints
  - Put the constraints in words first

**Excel Solution** 

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### FAQ. Do the units matter? How does one choose the units?

- The units do not matter so long as one is careful to use units correctly. It would be possible to have x<sub>A</sub> be in \$ millions and for x<sub>B</sub> to be in dollars.
- But some choices of units are more natural than others, and easier to use and to communicate.

### Generalizing the model

- Suppose that there are n investments, over m time periods.
- The payback of \$1 of investment j in period i is p<sub>ij</sub>.
   If investment j starts in period i, then p<sub>ij</sub> = -1, indicating that \$1 is invested in that period.
- Everything is reinvested.
- Maximize the total return in period m.
- Work with your partner on formulating the generalization.

### **Enrichments of the model**

 Finance concentrators: have we made assumptions that you would like to challenge? Can we deal with a more realistic model? **Scheduling Postal Workers** 

 Each postal worker works for 5 consecutive days, followed by 2 days off, repeated weekly.

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

 Minimize the number of postal workers (for the time being, we will permit fractional workers on each day.)

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### Formulating as an LP

- Select the decision variables
  - Let x<sub>1</sub> be the number of workers who start working on Sunday, and work till Thursday
  - Let  $\mathbf{x}_2$  be the number of workers who start on Monday ...
  - Let  $x_3$ ,  $x_4$ , ...,  $x_7$  be defined similarly.
- Work with your partner to formulate this LP.

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#### On the selection of decision variables

- Would it be possible to have y<sub>j</sub> be the number of workers on day j?
- It would be easy to formulate the constraint that the number of workers on day j is at least dj. How would one formulate the constraint that each worker works 5 days on followed by 2 days off.
- Conclusion: sometimes the decision variables are chosen to incorporate constraints of the problem. (more on homework 1).

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### Some Enhancements of the Model

- Suppose that there was a pay differential. The cost of workers who start work on day j is c<sub>j</sub> per worker.
- Suppose that one can hire part time workers (one day at a time), and that the cost of a part time worker on day j is PT<sub>i</sub>.

### **Another Enhancement**

- Suppose that the number of workers required on day j is d<sub>j</sub>. Let y<sub>j</sub> be the number of workers on day j.
- What is the minimum cost schedule, where the "cost" of having too many workers on day j is f<sub>i</sub>(y<sub>i</sub> - d<sub>i</sub>), which is a non-linear function?
- NOTE: this will lead to a non-linear program, not a linear program.

### Other enhancements

- Are there any other enhancements that you may think of with respect to workforce scheduling?
- If so, can we incorporate the enhancement into the model.

**Summary** 

- Gauss-Jordan solving of equations and other background in linear algebra
- A financial problem
- A problem in workforce scheduling.
- Note: Modeling in practice is an art form.
   It requires finding the right simplifications of reality for a given situation.

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