15.053 February 7, 2002

- A brief review of Linear Algebra
- Linear Programming Models

Handouts: Lecture Notes

1

Review of Linear Algebra

- Some elementary facts about vectors and matrices.
- The Gauss-Jordan method for solving systems of equations.
- Bases and basic solutions and pivoting.

2

Elementary Facts about Vectors

$v = [v_1 \quad v_2 \quad v_3 \quad v_4]$ is called a ***row vector***.

The ***transpose*** of v is a ***column vector***. $v^t = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$w = [w_1 \quad w_2 \quad w_3 \quad w_4]$ is another row vector.

The ***inner product*** of vectors w and v is given by: $v \circ w = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$

3

Matrix Multiplication

$A = (a_{ij}) \qquad B = (b_{ij}) \qquad C = (c_{ij}) = A \times B$

Suppose that A has n columns and B has n rows.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

4

Multiplying Matrices

Let C = (c_{ij}) = AB. Then c_{ij} is the *inner product* of row i of A and column j of B.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

For example, what is c_{23}?

$$c_{23} = a_{21} b_{13} + a_{22} b_{23} + a_{23} b_{33}$$

5

Multiplying Matrices

Let C = (c_{ij}) = A × B. Then each column of C is obtained by adding multiples of columns of A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \times \begin{bmatrix} 100 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 123 \\ 456 \\ 789 \end{bmatrix}$$

$$= 100 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 10 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Similarly, each row of C is obtained by adding multiples of rows of B.

6

Elementary Facts about Solving Equations

Solve for $Ax = b$, where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \end{bmatrix}_{2\times 3} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}_{2\times 1}$$

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 6 \end{aligned}$$

Find a linear combination of the columns of A that equals b.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ -1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Solving a System of Equations

x1	x2	x3	x4		
1	2	4	1	=	0
2	1	-1	-1	=	6
-1	1	2	2	=	-3

To solve a system of equations, use Gauss-Jordan elimination.

The system of equations

x_1	x_2	x_3	x_4		
1	2	4	1	=	0
2	1	-1	-1	=	6
-1	1	2	2	=	-3

Pivot on the element in row 1 column 1

x_1	x_2	x_3	x_4		
1	2	4	1	=	0
0	-3	-9	-3	=	6
0	3	6	3	=	-3

Subtract 2 times constraint 1 from constraint 2. Add constraint 1 to constraint 3.

Pivot on the element in Row 2, Column 2

x_1	x_2	x_3	x_4		
1	0	-2	-1	=	4
0	1	3	1	=	-2
0	0	-3	0	=	3

Divide constraint 2 by -3. Subtract multiples of constraint 2 from constraints 1 and 3.

Pivot on the element in Row 3, Column 3

x_1	x_2	x_3	x_4		
1	0	0	-1	=	2
0	1	0	1	=	1
0	0	1	0	=	-1

Divide constraint 3 by -3. Add multiples of constraint 3 to constraints 1 and 2.

What is a solution to this system of equations?

The fundamental operation: pivoting

x_1	x_2	x_3	x_4		
a_{11}	a_{12}	a_{13}	a_{14}	=	b_1
a_{21}	a_{22}	a_{23}	a_{24}	=	b_2
a_{31}	a_{32}	a_{33}	a_{34}	=	b_3

Pivot on a_{23}

Pivot on a_{23}

$\bar{a}_{11} = a_{11} - a_{13}(a_{21}/a_{23})$

x_1	x_2	x_3	x_4		
$\bar{a}_{11}$	a_{12}	0	a_{14}	=	b_1
a_{21}/a_{23}	a_{22}/a_{23}	1	a_{24}/a_{23}	=	b_2/a_{23}
a_{31}	a_{32}	0	a_{34}	=	b_3

What will be the next coefficient of b_1? a_{32}? of a_{ij} for $i \neq 2$?

Jordan Canonical Form for an m x n matrix

x_1	x_2	x_3	x_4		
1	0	0	-1	=	2
0	1	0	1	=	1
0	0	1	0	=	-1

There are m columns that have been transformed into unit vectors, one for each row. The variables in these columns are called "basic."

The "basic" solution is $x_1 = 2$, $x_2 = 1$, $x_3 = -1$ $x_4 = 0$

There is an easily determined solution for every choice of non-basic variables.

x_1	x_2	x_3	x_4		
1	0	0	-1	=	2
0	1	0	1	=	1
0	0	1	0	=	-1

The remaining variable x_4 is called non-basic.

If we set $x_4 = 2$, what solution do we get?

If we set $x_4 = \Delta$, what solution do we get?

Another Jordan Canonical Form for the same system of equations

x_1	x_2	x_3	x_4		
0	1	0	1	=	1
1	1	0	0	=	3
0	0	1	0	=	-1

What are the "basic"variables?

What is the basic solution?

Applications

- **A Financial Model**
- **Scheduling Postal Workers**

A Financial Problem

- Sarah has $1.1 million to invest in five different projects for her firm.
- Goal: maximize the amount of money that is available at the beginning of 2005.
  - (Returns on investments are on the next slide).
- At most $500,000 in any investment
- Can invest in CDs, at 5% per year.

Return on investments (undiscounted dollars)

	A	B	C	D	E
Jan. 2002	-1	-	-1	-1	-
Jan. 2003	.4	-1	1.2	-	-
Jan. 2004	.8	.4	-	-	-1
Jan. 2005	-	.8	-	1.5	1.2

Formulate Sarah's problem as an LP

- Payback for A: for every dollar invested in January of 2002, Sarah receives $.40 in January of 2003 and $.80 in January of 2004.
- FORMULATION.
  - STEP 1. Choose the decision variables
  - Let x_A denote the amount in millions of dollars invested in A.
    Define x_B, x_C, x_D, and x_E similarly.
  - Let x_2 denote the amount put in a CD in 2002. (Define x_3 and x_4 similarly)

Formulating the model

- With your partner formulate the LP model.
- Step 2. Formulate the objective function
  - put the objective function in words first. E.G. we are "minimizing cost" or "maximizing utility"
- Step 3. Formulate the constraints
  - Put the constraints in words first

Excel Solution

FAQ. Do the units matter? How does one choose the units?

- The units do not matter so long as one is careful to use units correctly. It would be possible to have x_A be in $ millions and for x_B to be in dollars.
- But some choices of units are more natural than others, and easier to use and to communicate.

Generalizing the model

- Suppose that there are n investments, over m time periods.
- The payback of $1 of investment j in period i is p_{ij}. If investment j starts in period i, then $p_{ij} = -1$, indicating that $1 is invested in that period.
- Everything is reinvested.
- Maximize the total return in period m.

- Work with your partner on formulating the generalization.

Enrichments of the model

- Finance concentrators: have we made assumptions that you would like to challenge? Can we deal with a more realistic model?

Scheduling Postal Workers

- Each postal worker works for 5 consecutive days, followed by 2 days off, repeated weekly.

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

- Minimize the number of postal workers (for the time being, we will permit fractional workers on each day.)

Formulating as an LP

- Select the decision variables
  - Let x_1 be the number of workers who start working on Sunday, and work till Thursday
  - Let x_2 be the number of workers who start on Monday …
  - Let x_3, x_4, …, x_7 be defined similarly.
- Work with your partner to formulate this LP.

On the selection of decision variables

- Would it be possible to have y_j be the number of workers on day j?
- It would be easy to formulate the constraint that the number of workers on day j is at least d_j. How would one formulate the constraint that each worker works 5 days on followed by 2 days off.
- Conclusion: sometimes the decision variables are chosen to incorporate constraints of the problem. (more on homework 1).

Some Enhancements of the Model

- Suppose that there was a pay differential. The cost of workers who start work on day j is c_j per worker.
- Suppose that one can hire part time workers (one day at a time), and that the cost of a part time worker on day j is PT_j.

Another Enhancement

- Suppose that the number of workers required on day j is d_j. Let y_j be the number of workers on day j.
- What is the minimum cost schedule, where the "cost" of having too many workers on day j is - $f_j(y_j – d_j)$, which is a non-linear function?
- NOTE: this will lead to a non-linear program, not a linear program.

Other enhancements

- Are there any other enhancements that you may think of with respect to workforce scheduling?
- If so, can we incorporate the enhancement into the model.

Summary

- Gauss-Jordan solving of equations and other background in linear algebra
- A financial problem
- A problem in workforce scheduling.

- Note: Modeling in practice is an art form. It requires finding the right simplifications of reality for a given situation.