

15.053

February 21, 2002

• Simplex Method Continued

Handouts: Lecture Notes

Note: This lecture is intended to be viewed as a slide show

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Today's Lecture

- Review of the simplex algorithm.
- Formalizing the approach
- Degeneracy and Alternative Optimal Solutions
- Is the simplex algorithm finite? (Answer, yes, but only if we are careful)

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LP Canonical Form =
LP Standard Form + Jordan Canonical Form

-z	x ₁	x ₂	x ₃	x ₄	
1	-3	2	0	0	= 0
0	-3	3	1	0	= 6
0	-4	2	0	1	= 2

z is not a decision variable

The basic variables are x₃ and x₄.

The non-basic variables are x₁ and x₂.

The basic feasible solution is x₁ = 0, x₂ = 0, x₃ = 6, x₄ = 2

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Basic Variables Non-basic Variables

-z	x ₁	x ₂	x _r	x _m	x _{m+1}	x _s	x _n	CV
1	0	0	0	0	\bar{c}_{m+1}	\bar{c}_s	\bar{c}_n	= $-\bar{z}_0$
0	1	0	0	0	$\bar{a}_{1,m+1}$	$\bar{a}_{1,s}$	$\bar{a}_{1,n}$	= \bar{b}_1
0	0	1	0	0	$\bar{a}_{2,m+1}$	$\bar{a}_{2,s}$	$\bar{a}_{2,n}$	= \bar{b}_2
0	0	0	1	0	$\bar{a}_{r,m+1}$	$\bar{a}_{r,s}$	$\bar{a}_{r,n}$	= \bar{b}_r
0	0	0	0	1	$\bar{a}_{m,m+1}$	$\bar{a}_{m,s}$	$\bar{a}_{m,n}$	= \bar{b}_m

m constraints, n variables

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Notation

- n number of variables
- m number of constraints
- s index of entering variable
- r index of pivot row
 - note: the rth basic variable leaves the basis
- The original data is c_j, a_{ij}, b_i
- After performing pivots we represent the revised coefficients as \bar{c}_j , \bar{a}_{ij} , \bar{b}_i

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The basic feasible solution

- The current values are all non-negative.
 - This is needed for canonical form
- There is a basic variable associated with each constraint.
 - in this case, the basic variable associated with constraint i is x_i.
- There are n-m nonbasic variables.
 - In this case, the nonbasic variables are x_{m+1}, ..., x_n.
- The bfs is as follows: x₁ = \bar{b}_1 , ..., x_m = \bar{b}_m . All other variables are 0.

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Optimality Conditions (maximization)

BV	-z	x ₁	x ₂	x ₃	x ₄	
-z	1	-2	-4	0	0	= -8
x ₃	0	-3	3	1	0	= 6
x ₄	0	-4	2	0	1	= 2

This basic feasible solution is optimal!

What are the optimality conditions, expressed in terms of \bar{c} ?

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Pivoting and the min ratio rule

Pivot in variable x_s , where $\bar{c}_s > 0$.

$$x_3 = 0 \text{ when } \Delta = 6/3$$

$$x_4 = 0 \text{ when } \Delta = 2/2.$$

BV	-z	x ₁	x ₂	x ₃	x ₄	
-z	1	-3	2	0	0	= 0
x ₃	0	-3	3	1	0	= 6
x ₄	0	-4	2	0	1	= 2

$$x_1 = 0$$

$$x_2 = \Delta$$

$$x_3 = 6 - 3\Delta$$

$$x_4 = 2 - 2\Delta$$

$$z = 2\Delta$$

Δ is set to the $\min(6/3, 2/2) = \min(\bar{b}_1/\bar{a}_{1s}, \bar{b}_2/\bar{a}_{2s})$.

The constraint with a changed basic variable is constraint r , where $r = \arg\min(\bar{b}_1/\bar{a}_{1s}, \bar{b}_2/\bar{a}_{2s}) = 2$. **Min ratio rule.**

Express the min ratio rule using general coefficients

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Pivoting to obtain a better solution

BV	-z	x ₁	x ₂	x ₃	x ₄	
-z	1	1	0	0	-1	= -2
x ₃	0	3	0	1	-3/2	= 3
x ₄	0	-2	1	0	1/2	= 1

$$x_1 = 0$$

$$x_2 = 1$$

$$x_3 = 3$$

$$x_4 = 0$$

$$z = 2$$

Pivot in variable x_s , where $\bar{c}_s > 0$.

Pivot out the basic variable for constraint r according to the min ratio rule.

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Alternative Optima (maximization)

-z	x ₁	x ₂	x ₃	x ₄	
1	0	-4	0	0	= -8
0	1	3	1	0	= 6
0	-1	2	0	1	= 2

There may be alternative optima if $\bar{c}_j \leq 0$ for all j and $\bar{c}_j = 0$ for some j where x_j is non-basic

This basic feasible solution is optimal!
Is there another optimal solution?

Suppose that x_1 entered the basis. What would leave?

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Alternative Optima (maximization)

-z	x ₁	x ₂	x ₃	x ₄	
1	0	-4	0	0	= -8
0	1	3	1	0	= 6
0	0	5	1	1	= 8

Let x_1 enter the basis. Let the basic variable in constraint 1 leave the basis.

Note: the solution is different, but the objective value is the same.

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Unboundedness

Express unboundedness in terms of algebraic notation.

-z	x ₁	x ₂	x ₃	x ₄	
1	1	0	0	-1	= -2
0	-3	0	1	-1.5	= 3
0	-2	1	0	.5	= 1

$$x_1 = \Delta$$

$$x_2 = 1 + 2\Delta$$

$$x_3 = 3 + 3\Delta$$

$$x_4 = 0$$

$$z = 2 + \Delta$$

The entering variable is x_1 .

Set $x_1 = \Delta$ and $x_4 = 0$.

If the coefficients in the entering column are ≤ 0 , then the solution is unbounded from above

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Review of notation

A basic feasible solution is optimal if $\bar{c}_j \leq 0$ for all j .

Assumption: the entering variable is x_s (and so $\bar{c}_s > 0$)

Pivot out the basic variable in row r , where $r = \operatorname{argmin}_i \{ \bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0 \}$, and thus $\bar{b}_r / \bar{a}_{rs} = \min \{ \bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0 \}$.

If $\bar{a}_{is} \leq 0$ for all i , then the solution is unbounded.

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Simplex Method (Max Form)

Step 0. The problem is in canonical form and $\bar{b} \geq 0$.

Step 1. If $\bar{c} \leq 0$ then stop. The solution is optimal. If we continue, then there exists some $\bar{c}_j > 0$.

Step 2. Choose any non-basic variable to pivot in with $\bar{c}_j > 0$, e.g., $\bar{c}_s = \max_j \{ \bar{c}_j : \bar{c}_j > 0 \}$. If $\bar{a}_{is} \leq 0$ for all i , then stop; the LP is unbounded. If we continue, then there exists some $\bar{a}_{rs} > 0$.

Step 3. Pivot out the basic variable in row r , where r is chosen by the min ratio rule, that is $r = \operatorname{argmin}_i (\bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0)$.

Step 4. Replace the basic variable in row r with variable x_s and re-establish canonical form (i.e., pivot on the coefficient \bar{a}_{rs} .)

Step 5. Go to Step 1.

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Preview of what is to come

- Degeneracy and improving solutions
- Proving finiteness and optimality under no degeneracy
- Handling degeneracy
- Obtaining an initial canonical form

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Degeneracy

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	0	0	=	-2
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	0

$x_1 = 0$
 $x_2 = \Delta$
 $x_3 = 6 - 3\Delta$
 $x_4 = 0 - 2\Delta$
 $z = 2 + 2\Delta$

A bfs is degenerate if $\bar{b}_i = 0$ for some i .

Otherwise, it is non-degenerate.

Suppose that variable x_2 will enter the basis.

Degenerate \Rightarrow the solution may stay the same
 $\Rightarrow z$ stays the same

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A degenerate pivot

BV	-z	x_1	x_2	x_3	x_4		
-z	1	1	0	0	-1	=	-2
x_3	0	3	0	1	-3/2	=	6
x_4	0	-2	1	0	1/2	=	0

$x_1 = 0$
 $x_2 = \Delta$
 $x_3 = 6$
 $x_4 = 0$
 $z = 2$

The entering variable is x_2 . The exiting variable is the one in constraint 2.

In this case the solution did not change.
 (But the tableau did.)

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Non-degeneracy leads to strict improvement

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	0	0	=	-2
x_3	0	-3	3	1	0	=	6
x_4	0	-4	2	0	1	=	2

$x_1 = 0$
 $x_2 = \Delta$
 $x_3 = 6 - 3\Delta$
 $x_4 = 2 - 2\Delta$
 $z = 2 + 2\Delta$

If the bfs is non-degenerate, then the entering variable can strictly increase, and the objective value will strictly improve.

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Theorem. If every basic feasible solution is non-degenerate, the simplex method is finite.

1. The number of basic feasible solutions. is at most $n! / (n-m)! m!$, which is the number of ways of selecting m basic variables
 2. Each basic feasible solution is different
 - Each has a better cost than the last, assuming non-degeneracy
- Therefore, the simplex method is finite

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Handling Degeneracy

- Perturb the RHS just a little, in just the right way
 - No basis is degenerate
 - Every bfs to the perturbed problem is also a bfs for the original problem
 - The optimal basis for the perturbed problem is optimal for the original problem

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An example of a perturbed initial bfs

BV	-z	x_1	x_2	x_3	x_4	
-z	1	-3	2	0	0	= 0
x_3	0	-3	3	1	0	= 6.000000000000013
x_4	0	-4	2	0	1	= 2.000000000000041

In the original bfs, $x_3 = 6$, $x_4 = 2$.

If the perturbation is small enough, then each bfs for the perturbed problem is a bfs for the original problem

If the perturbation is chosen carefully, the simplex method is non-degenerate and thus finite.

Some Remarks on Degeneracy and Alternative Optima

- It might seem that degeneracy would be rare. After all, why should we expect that the RHS would be 0 for some variable?
- In reality, degeneracy is incredibly common.
- The simplex method is not necessarily finite unless care is taken in the pivot rule. In reality, almost no one takes care, and the simplex method is not only finite, but it is incredibly efficient.
- The issue of alternative optima arises frequently, and is of importance in practice.

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How does one obtain an initial bfs?

-z	x_1	x_2	x_3	x_4	
1	-3	2	4	1	= 0
0	-3	3	2	5	= 6
0	-4	2	1	3	= 2

FACT: Obtaining an initial basic feasible solution is (theoretically) as difficult as obtaining an optimal one

IDEA: Use the simplex algorithm to obtain an initial bfs.

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Reducing to a Previously Solved Problem

Juan's Problem

Find a feasible solution to

$$-3x_1 + 3x_2 + 2x_3 + 5x_4 = 6$$

$$-4x_1 + 2x_2 + 1x_3 + 3x_4 = 2$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4$$

Maria's Problem

Minimize $x_5 + x_6$

subject to:

$$-3x_1 + 3x_2 + 2x_3 + 5x_4 + x_5 = 6$$

$$-4x_1 + 2x_2 + 1x_3 + 3x_4 + x_6 = 2$$

$$x_j \geq 0 \text{ for } j = 1, 2, 3, 4, 5, 6$$

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Are Juan's and Maria's Problems equivalent?

- If there is a feasible solution to Juan's problem, then this feasible solution solves Maria's problem.
e.g., $(1 \ 3 \ 0 \ 0)$ is feasible for Juan
 $(1 \ 3 \ 0 \ 0 \ 0 \ 0)$ is optimal for Maria
- If the best solution to Maria's problem has objective 0, then this solves Juan's problem.
- For maximization, Maria's problem is
maximize $w = -x_5 - x_6$

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Maria's Problem in Tableau Form

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6	
-w	1	0	0	0	0	-1	-1	= 0
x_5	0	-3	3	2	5	1	0	= 6
x_6	0	-4	2	1	3	0	1	= 2

This problem is not yet in canonical form. But it is close.

What transformation do we need to do?

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Maria's Problem in Canonical Form

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6	
-w	1	-7	5	3	8	0	0	= 8
x_5	0	-3	3	2	5	1	0	= 6
x_6	0	-4	2	1	3	0	1	= 2

Then run the simplex algorithm on Maria's problem

It will find a feasible solution (if one exists) for Juan's problem, and it will terminate with a bfs.

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The optimal basis for Maria's Problem

BV	-w	x_1	x_2	x_3	x_4	x_5	x_6	
-w	1	0	0	0	0	-1	-1	= 0
x_5	0	1	0	1/6	1/6	1/3	-1/2	= 1
x_6	0	0	1	5/6	11/6	2/3	-1/2	= 3

At the end, one can obtain a bfs for the original problem by dropping x_5 and x_6 .

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Recovering a bfs for the original problem

BV	-z	x_1	x_2	x_3	x_4		
-z	1	-3	2	4	1		= 0
x_5	0	1	0	1/6	1/6		= 1
x_6	0	0	1	5/6	11/6		= 3

At the end, eliminate the "artificial variables" x_5 and x_6 .

Reintroduce the original objective

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Recovering a bfs for the original problem

BV	-z	x_1	x_2	x_3	x_4		
-z	1	0	0	17/6	-13/6		= -3
x_5	0	1	0	1/6	1/6		= 1
x_6	0	0	1	5/6	11/6		= 3

At the end, eliminate the "artificial variables" x_5 and x_6 .

Reintroduce the original objective

Then bring into canonical form.

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Summary

- Maria's problem is referred to as "Phase 1" and the additional variables are referred to as "artificial variables."
- One can create the "Phase 1" problem directly, without needing an interpretation.
- Solving the phase 1 problem either
 1. shows that there is no feasible solution to the original problem or else
 2. it ends with a bfs for the original problem

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Creating a Phase 1 Problem

BV	-w	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	
-w	1	0	0	0	0	-1	-1	= 0
x ₅	0	-3	3	2	5	1	0	= 6
x ₆	0	-4	2	1	3	0	1	= 2

Eliminate the objective function, for now

Add the artificial variables

Minimize the sum of the artificials

Bring into canonical form

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Creating a Phase 1 Problem

BV	-w	x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	
-w	1	-7	5	3	8	0	0	= 8
x ₅	0	-3	3	2	5	1	0	= 6
x ₆	0	-4	2	1	3	0	1	= 2

Eliminate the objective function, for now

Add the artificial variables

Minimize the sum of the artificials

Bring into canonical form

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Creating a bfs from the phase 1 solution

BV	-z	x ₁	x ₂	x ₃	x ₄	
-z	1	0	0	17/6	-13/6	= -3
x ₅	0	1	0	1/6	1/6	= 1
x ₆	0	0	1	5/6	11/6	= 3

At end, if $w > 0$, report that there is no feasible solution.

If $w = 0$, eliminate artificial variables (or keep the columns but forbid them from pivoting in).

Reintroduce the original objective

Then bring into canonical form.

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Potential Difficulties

- If the original problem is degenerate, it is possible that an artificial is in the basis at the end of phase 1.
 - Solution: pivot out the artificial variable
- If the original problem has a redundant constraint, then it is possible that an artificial variable is in the basis at the end of phase 1, and cannot be pivoted out.
 - Solution: eliminate (or ignore) the redundant constraint

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Summary

- Review of the simplex algorithm.
- Degeneracy and Alternative Optimal Solutions
- Is the simplex algorithm finite? (Answer, yes, but only if we are careful)
- How do we get an initial BFS?
 - Phase I Approach

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