IE513 Homework 1 Solutions

1. One halfspace contains the other halfspace when the two halfspaces are parallel. The conditions under which

$$\{x \mid a^T x \leq b\} \subseteq \{x \mid \tilde{a}^T x \leq \tilde{b}\}$$

$a \neq 0, \tilde{a} \neq 0$ are

- a and $\tilde{a}$ must be parallel, i.e., $a = \alpha\tilde{a}$ for any $\alpha \in \mathcal{R}^+$
- $b \leq \alpha\tilde{b}$

When $b = \alpha\tilde{b}$ the two halfspaces are equal.

2. The distance between the two hyperplanes is $|b_1 - b_2|/\|a\|$. To see this, choose two points x_1 and x_2 on the hyperplanes and take the vectors through the origin to these points that are in the same direction of the normal vector a. The distance between the two hyperplanes is also the distance between the two points x_1 and x_2 where the hyperplanes intersect the line through the origin and parallel to the normal vector a. These points can be represented as

$$x_1 = (c_1/\|a\|)a, \qquad x_2 = (c_2/\|a\|)a$$

where $a/\|a\|$ is the unit vector along a. In order to find c_1 and c_2,

$$a^T x_1 = b_1 \Rightarrow a^T c_1/\|a\|a = b_1 \Rightarrow c_1 = b_1/\|a\|.$$

Similarly, $c_2 = b_2/\|a\|$. Then, these points are given by

$$x_1 = (b_1/\|a\|^2)a, \qquad x_2 = (b_2/\|a\|^2)a,$$

and the distance is

$$\|x_1 - x_2\| = |b_1 - b_2|/\|a\|.$$

3. A *polyhedron* is defined as the solution set of a finite number of linear equalities and inequalities, i.e.,

$$\mathcal{P} = \{x \mid Ax \leq b,\ Cx = d\}.$$

A polyhedron is thus the intersection of a finite number of halfspaces and hyperplanes.
A set $C \subseteq \mathcal{R}^n (S \neq \emptyset)$ is called a *subspace* if for any $x_1, x_2 \in C$ and $\alpha,\ \beta \in \mathcal{R}$, we have

$$\alpha x_1 + \beta x_2 \in C.$$

A set $C \subseteq \mathcal{R}^n$ is *affine* if the line through any two distinct points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and $\theta \in \mathcal{R}$, we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$

Affine $\Rightarrow$ Convexity
Any subspace is affine, hence convex.

a)

$$y_1 a_1 + y_2 a_2 \leq a_1 + a_2$$
$$-y_1 a_1 - y_2 a_2 \leq a_1 + a_2$$
$$y_1 \leq 1$$
$$-y_1 \leq 1$$
$$y_2 \leq 1$$
$$-y_2 \leq 1$$

Since, we can write S as an intersection of finitely many halfspaces and hyperplanes, S is a polyhedron. Is it a subspace?
Let $z_1 = y_1 a_1 + y_2 a_2$ and $z_1 = y_1' a_1 + y_2' a_2$ be two elements in S. Check whether $\alpha z_1 + \beta z_2 \in S$ or not for any $\alpha,\ \beta \in \mathcal{R}$. Now, $\alpha z_1 + \beta z_2 = \alpha(y_1 a_1 + y_2 a_2) + \beta(y_1' a_1 + y_2' a_2)$. A choice of $\alpha = 1,\ \beta = 1,\ y_1 = y_2 = y_1' = y_2' = 1$, implies $\alpha z_1 + \beta z_2 = 2a_1 + 2a_2$ which is not an element of S. So, S is not a subspace.
A similar argument shows that S is not affine either.

b) S is a polyhedron. In fact the definition involves linear inequalities $x \geq 0$ and three equality constraints, so we only have to write the equality constraints as inequalities. This yields a set of $n+6$ inequalities:

$$-x_i \leq 0, \quad i = 1, \ldots, n$$
$$\mathbf{1}^T x \leq 1$$
$$-\mathbf{1}^T x \leq -1$$
$$\sum_i a_i x_i \leq b_1$$
$$-\sum_i a_i x_i \leq -b_1$$
$$\sum_i a_i^2 x_i \leq b_2$$
$$-\sum_i a_i^2 x_i \leq -b_2$$

c) Clearly S is not a polyhedron for $n = 2$ and general n. So, although we define S as an intersection of halfspaces, it is not a polyhedron, since we need infinitely many halfspaces.

d) For general n, we can show that S is the intersection of the set $\{x \mid |x_k| \leq 1, \ \ k = 1, \ldots, n\}$ and the nonnegative orthant $\{x \mid x \geq 0\}$. This follows from the following fact:

$$|x_i| \leq 1, i = 1, \ldots, n \Longleftrightarrow x^T y \leq 1 \ \text{ for all } y \text{ with } \ \sum_i |y_i| = 1$$

All this implies that we can describe S by a finite number of linear inequalities: it is the intersection of non-negative orthant with the set $\{x \mid -1 \leq x \leq 1\}$, i.e., the solution of $2n$ linear inequalities

$$-x_i \leq 0 \ \ i = 1, \ldots, n$$

$$x_i \leq 1, \ \ i = 1, \ldots, n.$$

Note that as in problem (c) the set S was given as an intersection of an infinite number of halfspaces. The difference is that here most of the linear inequalities are redundant, and only a finite number are needed to characterize S.

e) This set is a polyhedron (in fact, a halfspace). By squaring both sides of the inequality $\|x - x_0\| \leq \|x - x_1\|$, we obtain the equivalent condition

$$\|x - x_0\|^2 \leq \|x - x_1\|^2 \Longleftrightarrow x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2x_1^T x + x_1^T x_1$$

$$\Longleftrightarrow 2(x_1 - x_0)^T x \leq \|x_1\|^2 - \|x_0\|^2,$$

which is a linear inequality.

f) Referring to (e), we see that S is the intersection of K halfspaces, i.e., a polyhedron, and that it can be represented by the K inequalities

$$2(x_k - x_0)^T x \leq \|x_k\|^2 - \|x_0\|^2, \ \ k = 1, \ldots, K.$$

None of these sets are affine sets or subspaces, except in some trivial cases. For example, the set defined in (b) is an affine set if $n = 1$ and $S = \{1\}$; etc.

4. In order for the output y to be equal to 1, the inequality $-x_1 + 2x_2 \geq -1$ and at least one of the other two inequalities must hold. So, our solution space makes up all the $\mathcal{R}^2$ except the region defined by these three inequalities: $-x_1 + 2x_2 \leq -1, -x_1 + x_2 \leq 1, -x_1 - x_2 \leq -1$

5. Since $y_i = a_i^T x + v_i$ with $|v_i| < \alpha$, then

$$a_i^T x - y_i \leq \alpha$$

$$-a_i^T x + y_i \leq \alpha \ \text{ for } \ i = 1, \ldots, k.$$

So, X can be represented by an intersection of $2K$ number of hyperplanes, thus a polyhedron.
For the second part, if some measurements are faulty, we lose the corresponding $2J$ number of inequalities. The remaining $2(K - J)$ number of inequalities still forms a polyhedron.

6. Optimal value and the optimal set for the following values of c:

- $c = (-1, 0, 1) : x^* = (3, 0, 0),\ p^* = -3$
- $c = (0, 1, 1) : X_{opt} = \{x \in \mathcal{R}^3 \mid 1 \leq x_1 \leq 3,\ x_2 = 0,\ x_3 \geq 0\}$ and $p^* = -0$
- $c = (0, 0, -1) : X_{opt} = \emptyset,\ p^* = -\infty$

7. A point x is said to be an extreme point of $\mathcal{P}$ if x lies on some n linearly independent defining hyperplanes of $\mathcal{P}$.

- The first 4 of 5 inequalities for $\tilde{x} = (1, 1, 1, 1)$ are binding(active) and the matrix A corresponding to the active constraints has rank 4. So, $\tilde{x}$ is an extreme point.
  As vector c we can choose any (strictly) negative combination of the rows of matrix A, e.g., $c = (8, 16, 4, -15)$
- In this case, the matrix A has rank 3, so x is not an extreme point.

8. Suppose $\hat{y}$ and $\hat{z}$ are a solution of (2). Thus $A\hat{y} - \hat{z}b \leq 0,\ d^T\hat{y} + \delta\hat{z} = 1$ and $\hat{z} \geq 0$ holds. If $\hat{z} = 0$, then $A\hat{y} \leq 0$ which implies $\hat{y}$ is a recession direction of $\mathcal{P}$ which contradicts the fact that $\mathcal{P}$ is bounded. So, $\hat{z} > 0$. Then,

$$A\hat{y} - \hat{z}b \leq 0 \Rightarrow A\hat{y} \leq \hat{z}b \Rightarrow A\hat{y}/\hat{z} \leq b.$$

Define $\hat{x} = \hat{y}/\hat{z}$. Then $A\hat{x} \leq b$ which implies $\hat{x}$ is a solution of (1).

9. a) Assume (without loss of generality) that the components of c are sorted in decreasing order:

$$c_1 \geq c_2 \geq \ldots \geq c_n.$$

The optimal value of the LP is

$$p^* = c_1 + c_2 + \ldots + c_k,$$

i.e., the sum of the largest components of c. The optimal solution x^* has all components zero, except

$$x_1^* = x_2^* = \ldots x_k^* = 1.$$

(If $c_k = c_{k+1}$ then this is not the only optimal solution.)

b) As in (a), we assume the components of c are sorted in decreasing order. Define $\hat{k}$ as

$$\hat{k} = \max\{i \in \mathcal{Z} \mid i \leq k,\ c_i \geq 0\}$$

Then the optimal value is

$$p^* = c_1 + \cdots + c_{\hat{k}}$$

(the sum of the largest k positive components of c). The optimal x^* has all components zero except

$$x_1^* = \cdots = x_{\hat{k}}^* = 1.$$

c) Assume that the components of c are sorted in the order of decreasing absolute value, i.e.,

$$|c_1| \geq |c_2| \geq \ldots \geq |c_k|.$$

Then the optimal value is

$$p^* = |c_1| + \cdots + |c_k|$$

(the sum of the largest k absolute values of components of c). The optimal x^* has all components zero except

$$x_i^* = \ sign\ c_i,\ i = 1, \ldots, k,$$

The optimal y^* has components $y_i^* = |x_i^*|$.

d) Optimal solution is $v = 0,\ u = c$. Optimal value is $1^T c$.

e) Optimal solution is $v = 0,\ u = c$. Optimal value is $d_1^T c$.