IE513 Homework 2 Solutions

1. (Exercise 12)

(a) The problem is equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} y_i \\ \text{subject to} & -y \leq Ax - b \leq y \\ & -\mathbf{1} \leq x \leq \mathbf{1} \end{array}$$

The variables are $x \in I\!R^n$ and $y \in I\!R^m$.
In matrix notation, the problem is

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \leq \tilde{b} \end{array}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \ \tilde{c} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} A & -I \\ -A & -I \\ I & 0 \\ -I & 0 \end{bmatrix}, \ \tilde{b} = \begin{bmatrix} b \\ -b \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}.$$

(b) If we introduce a new variable $y \in I\!R^n$, we can express the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \leq x \leq y \\ & -\mathbf{1} \leq Ax - b \leq \mathbf{1} \end{array}$$

which is an LP in x and y.

Another good solution is to write x as the difference of two nonnegative vectors $x = x^+ - x^-$, and to express the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x^+ + \mathbf{1}^T x^- \\ \text{subject to} & -\mathbf{1} \leq Ax^+ - Ax^- - b \leq \mathbf{1} \\ & x^+ \geq 0, \ x^- \geq 0 \end{array}$$

which is an LP in $x^+ \in I\!R^n$ and $x^- \in I\!R^n$

(c) We can introduce new variables $y \in I\!R^m$ and $t \in I\!R$ and write the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y + t \\ \text{subject to} & -y \leq Ax - b \leq y \\ & -t\mathbf{1} \leq x \leq t\mathbf{1}, \end{array}$$

which is an LP in x, y and t.

2. (Exercise 13)

Here, our aim is to minimize the maximum deviation $\phi(p)$:

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,n} |I_k - I_{des}| \\ \text{subject to} & 0 \le p_i \le 1 \ \text{ for } \ i = 1, \ldots, m \end{array}$$

In other way

$$\begin{array}{ll} \text{minimize} & \|\phi(p)\|_\infty \\ \text{subject to} & 0 \le p_i \le 1 \ \text{ for } \ i = 1, \ldots, m \end{array}$$

After using the argument in Problem 1, the LP formulation of the problem becomes:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & I_k - I_{des} \le t\mathbf{1} \ \text{ for } \ k = 1, \ldots, n \\ & -I_k + I_{des} \le t\mathbf{1} \ \text{ for } \ k = 1, \ldots, n \\ & 0 \le p_i \le 1 \ \text{ for } \ i = 1, \ldots, m \end{array}$$

where $I_k = \sum_{j=1}^m a_{kj} p_j = a_k^T p$, for $k = 1, \ldots, n$.

3. (Exercise 15)

- Problem (b) is separable. We can minimize $\|Ax - b_x\|_1$ by solving the LP

  $$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & -s \le Ax - b_x \le s \end{array}$$

  with variables x, s. We minimize $\|Ay - b_y\|_1$ by solving the LP

  $$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & -s \le Ay - b_y \le s \end{array}$$

  (variables y, s). Of course you can solve both problems at once, using the LP

  $$\begin{array}{ll} \text{minimize} & \mathbf{1}^T v + \mathbf{1}^T w \\ \text{subject to} & -v \le Ax - b_x \le v \\ & -w \le Ay - b_y \le w \end{array}$$

  (variables v, w, x, y). The solution will be the same, but solving the larger problem will take more time than the two smaller problems separately.
- Problem (c) is not separable in a minimization over x and one over y. It is equivalent to the LP

  $$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -v \le Ax - b_x \le v \\ & -w \le Ay - b_y \le w \\ & v + w \le t\mathbf{1} \end{array}$$

  where the variables are v, w and t. The first two sets of inequality constraints are equivalent with $v_i \ge |a_i^T x - b_{x,i}|$ and $w_i \ge |a_i^T y - b_{y,i}|$ for $i = 1, \ldots, m$. The last constraints state that $t \ge \max_i(v_i + w_i)$, *i.e.*,

  $$t \ge \max_{i=1,\ldots,m} \left(|a_i^T x - b_{x,i}| + |a_i^T y - b_{y,i}| \right).$$

- Problem (d) is again separable. We can minimize $\sum_{i=1}^N h(a_i^T x - b_{x,i})$ via the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & -s \le Ax - b_x \le s \\ & s \ge \gamma \mathbf{1} \end{array}$$

  with variables s, x. The constraints state that $s_i \ge \max\{a_i^T x - b_{x,i}, -a_i^T x - b_{x,i}, \gamma\} = \max\{|a_i^T x - b_{x,i}|, \gamma\}$, $i = 1, \ldots, m$. The LP that determines y is similar.

Figure 1 and 2 show the results for the problem with $n = 100$. Table 1 compares the values of total wirelength $\sum_i |a_i^T x - b_i|$, the maximum wirelength $\max_i |a_i^T x - b_i|$, and the overlap.

We see that the ℓ^∞ objective spreads the cells over the largest interval, followed by the ℓ^2 objective and the ℓ^1 objective with dead zone. The ℓ^1 objective clusters all cells in a very narrow region (hence a very large overlap).

The overall conclusion is that ℓ^1 objective with dead zone provides a total wirelength that is not much worse than in the pure ℓ^1 solution, but with a significantly reduced overlap.

	total wirelength	max. wirelength	overlap
ℓ^2 solution	85	1.2	3 %
ℓ^1 solution	64	1.0	88 %
ℓ^∞ solution	98	1.0	3 %
ℓ^1 with dead zone	66	1.1	31 %

Table 1: Comparison of four solutions.

The histograms can be seen at the end.

4. (Exercise 16)
We can formulate the problem as follows:

$$\begin{array}{lll} \text{minimize} & t & \\ \text{subject to} & a^T v^i \le b + t & \text{for } i = 1, \ldots, k \\ & a^T w^i \le -b + t & \text{for } i = 1, \ldots, l \end{array}$$

If $t \le 0$, then the problem is feasible.

5. (Exercise 27)
An example with one variable and one constraint is

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0.x \le -1 \end{array}$$

and

$$\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & 0.z + 1 = 0 \\ & z \geq 0 \end{array}$$

6. (Exercise 28)
   Firstly, $\hat{x} = (1, 1, 1, 1)$ is primal feasible since it satisfies the constraints of the problem with objective value 64.

   The dual problem is

$$\begin{array}{ll} \text{maximize} & -3z_1 + 5z_2 - 8z_3 - 7z_4 + 4z_5 \\ \text{subject to} & -1z_1 - 1z_2 + 0z_3 - 6z_4 + 1z_5 = 47 \\ & -6z_1 - 2z_2 + 3z_3 - 11z_4 + 6z_5 = 93 \\ & 1z_1 + 7z_2 - 10z_3 - 2z_4 - 1z_5 = 17 \\ & 3z_1 + 1z_2 - 1z_3 + 12z_4 - 3z_5 = -93 \end{array}$$

   Since the 5th constraint in the primal problem is non-binding, by complementary slackness theorem, we can conclude that $z_5 = 0$. Then solving the dual problem for z yields the solution $z = (3, 2, 2, 7, 0)$ with an objective value 64. Then by the strong duality theorem $\hat{x}$ is the optimal solution.

7. (Exercise 29)
   (a) For $\hat{x} = (1, 1, 1, 1)$, the last four constraints are binding and the corresponding $\hat{A}$ matrix has full rank, i.e., $\text{rank}(\hat{A}) = 4$. Then we can conclude that $\hat{x}$ is an extreme point.

   (b) By the same argument in problem (6), we find a dual feasible solution $z = (0, 1, 2, 0, -3)$ with objective value 221. Then again by strong duality theorem we can conclude that $\hat{x}$ is optimal.

   (c) Alternative optimal solution exists since in the optimal solution one of the reduced costs is equal to zero.