IE513 Homework 3 Solutions

1. (Exercise 31)

(a) As it can be seen that x^* is primal feasible and z^* is dual feasible. The objective value of the x^* in primal problem is $p^* = c^T x^* = -76$ and the objective value of z^* in dual problem is $d^* = -b^T z^* = -76$. Then by strong duality theorem, both solutions are optimal.

(b) First 4 constraints of the problem are tight for $x^* = (1,1,1,1)$. So, $x^*(\epsilon) = x^*$. Then $p^*(\epsilon) = (c+\epsilon d)^T x^*(\epsilon) = (c+\epsilon d)^T x^* = p^* + \epsilon d^T x^*$. So, $p^*(\epsilon) = -76 - 21\epsilon$. The interval at which the ϵ is valid can be found from the dual problem: $A^T z + c + \epsilon d = 0$ and $z \geq 0$ implies that $-3 \leq \epsilon \leq 4$.

(c) Primal optimal solution is still $x^* = (1,1,1,1)^T$ since it does not depend on ϵ. Dual optimal solution is given by $A^T z + c + \epsilon d = 0$. Then

$$z^*(\epsilon) = \begin{bmatrix} 9 \\ 9 \\ 4 \\ 9 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}.$$

2. (Exercise 32)

(a) All constraints except the first are active at $x = (1,1,1,1)$, so complementary slackness implies that $z_1 = 0$ at the dual optimum. For this problem, the complementary slackness condition uniquely determines z:
we must have

$$\bar{A}^T \bar{z} + c = 0,$$

where

$$\bar{A} = \begin{bmatrix} -17 & 12 & 7 & 11 \\ 1 & 0 & -6 & 1 \\ 3 & 3 & 22 & -1 \\ -11 & 2 & -1 & -8 \end{bmatrix}, \quad \bar{z} \begin{bmatrix} z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}$$

$\bar{A}$ is nonsingular, so $\bar{A}^T \bar{z} + c = 0$ has a unique solution: $\bar{z} = (2,1,2,2)$. All components are nonnegative, so we conclude that $z = (0,2,1,2,2)$ is dual feasible.

(b) We expect that for small ϵ the same primal and dual constraints remain active. Let us first construct $x^*(\epsilon)$ and $z^*(\epsilon)$ under that assumption, and then verify using complementary slackness that they are optimal for the perturbed problem. To keep the last four constraints of $x^*(\epsilon)$ active, we must have

$$x^*(\epsilon) = (1,1,1,1) + \epsilon \Delta x$$

where $\bar{A}\Delta x = (f_2, f_3, f_4, f_5)$. We find $\Delta x = (0, 1, 2, -1$. $x^*(\epsilon)$ is primal feasible as long as

$$A((1,1,1,1) + \epsilon(0,1,2,-1) \leq b + \epsilon f.$$

By construction, this holds with equality for constraints 2-5. For the first inequality we obtain

$$7 + 7\epsilon \leq 8 + 6\epsilon.$$

i.e., $\epsilon \leq 1$.
If we keep the first component of $z^*(\epsilon)$ zero, the other components follow from, $A^T z^*(\epsilon) + c + \epsilon d = 0$. We must have

$$z^*(\epsilon) = (0, 2, 1, 2, 2) + \epsilon \Delta z$$

where $A^T \Delta z + f = 0$ and $\Delta z_1 = 0$. We find $\Delta z = (0, -1, 2, 0, 2)$. By construction, $z^*(\epsilon)$ satisfies the equality constraints $A^T z^*(\epsilon) + c + \epsilon f = 0$, so it is dual feasible if its components are nonnegative:

$$z^*(\epsilon) = (0, 2 - \epsilon, 1 + 2\epsilon, 2, 2 + 2\epsilon) \geq 0,$$

i.e., $-1/2 \leq \epsilon \leq 2$.
In conclusion, we constructed $x^*(\epsilon)$ and $z^*(\epsilon)$ that are primal and dual feasible for the perturbed problem, and complementary. Therefore they must be optimal for the perturbed problems in the interval $-1/2 \leq \epsilon \leq 2$.

(c) The optimal value is a quadratic function of ϵ:

$$p^*(\epsilon) = (c + \epsilon d)^T x^*(\epsilon) = -(b + \epsilon f)^T z^*(\epsilon) = -40 - 15\epsilon + 25\epsilon^2.$$

3. (Exercise 33)

(a) We can formulate the problem as the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{bmatrix} A & -I \\ -A & -I \\ 0 & \mathbf{1^T} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} -b \\ b \\ 1 \end{bmatrix} \end{array}$$

with variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

(b) The dual LP is

$$\begin{array}{ll} \text{maximize} & b^T u - b^T v - w \\ \text{subject to} & A^T u - A^T v + c = 0 \\ & -u - v + w\mathbf{1} = \mathbf{0} \\ & u \geq 1,\ v \geq 1,\ w \geq 1 \end{array}$$

with variables $u \in \mathbb{R}^m$, $v \in \mathbb{R}^m$ and $w \in \mathbb{R}$.
To prove that the above problem is equivalent to

$$
\begin{array}{ll}
\text{maximize} & b^T z - \|z\|_\infty \\
\text{subject to} & A^T z + c = 0,
\end{array}
$$

we first assume that z satisfies $A^T z + c = 0$. Define w, u and v as follows: $w = \|z\|_\infty$ and, for $i = 1, \ldots, n$,

$$u_i = \max z_i, 0 + \frac{1}{2}(w - |z_i|), \;\; v_i = \max -z_i, 0 + \frac{1}{2}(w - |z_i|).$$

It is easily verified that u, v, w satisfy the constraints of the former problem and that

$$b^T u - b^T v - w = b^T z - \|z\|_\infty.$$

From this we can conclude that the optimal value of the latter problem is less than or equal to the optimal value of former problem.
Conversely, assume u, v, w are feasible in former problem. Define $z = u - v$. By the first constraint in former problem, we have $A^T z + c = 0$. By the second constraint

$$|z_i| = |u_i - v_i| \leq u_i + v_i = w.$$

Therefore $\|z\|_\infty \leq w$, and we have

$$b^T z - \|z\|_\infty \geq b^T(u - v) - w.$$

Therefore the optimal value of latter problem is greater than or equal to the optimal value of former problem.
Combining the two results we can conclude that the optimal values are equal.

(c) Applying the inequality $v^T w \leq \|v\|_1 \|w\|_\infty$ to $v = Ax + b$ and $v = z$, we obtain

$$
\begin{array}{rcl}
c^T x & = & -z^T A x \\
& = & -z^T(Ax + b) + b^T z \\
& \geq & -\|z\|_\infty \|Ax + b\|_1 + b^T z \\
& \geq & -\|z\|_\infty + b^T z.
\end{array}
$$

The first equation follows from feasibility of z in the latter problem.

4. (Exercise 35)

(a) Assume that $a_{k-1} \leq \alpha \leq a_k$, then the LP formulation is

$$
\begin{array}{ll}
\text{maximize} & \sum_{i=k}^n p_i \\
\text{subject to} & \sum_{i=1}^n a_i p_i = b \\
& \sum_{i=1}^n p_i = 1 \\
& p_i \geq 0 \text{ for } i = 1, \ldots, n.
\end{array}
$$

(b) The dual problem when we write the primal problem as a minimization problem and define z as $z = \begin{bmatrix} -\lambda \\ -v \end{bmatrix}$ is :

$$\begin{array}{ll} \text{minimize} & \lambda b + v \\ \text{subject to} & \lambda a_i + v \geq 0 \text{ for all } a_i < \alpha \\ & \lambda a_i + v \geq 1 \text{ for all } a_i \geq \alpha \end{array}$$

Depending on the value of b we can obtain optimal solution by adjusting λ and v.
i) If $b \geq \bar{a}$ then $f(b) = 1$ satisfies all the constraints and it is the minimum.
ii) If $b \leq \bar{a}$ then

$$\frac{f(b)}{b - a_1} = \frac{1}{\bar{a} - a_1} \Rightarrow f(b) = \frac{b - a_1}{\bar{a} - a_1}.$$

Optimal values of λ and v when $b \leq \bar{a}$ can be found by $f(b) = \frac{1}{\bar{a}-a_1}b - \frac{a_1}{\bar{a}-a_1}$. So, $\lambda = \frac{1}{\bar{a}-a_1}$ and $v = \frac{-a}{\bar{a}-a_1}$. When $b \geq \bar{a}$, $f(b) = 1 = \lambda b + v$. So, $\lambda = 0$ and $v = 1$.

(c) i) If $b \geq \bar{a}$, then primal optimal value is 1 by strong duality, so $\sum_{i=k}^{n} p_i = 1$ and $p_i = 0$ for $i < k$. ii) If $b \leq \bar{a}$ there are two active constraints in the dual: $\lambda a_1 + v = 0$ and $\lambda \bar{a} + v = 1$. Then,

$$p_1 = \frac{b - \bar{a}}{a_1 - \bar{a}} \quad \text{and} \quad p_k = \frac{a_1 - b}{a_1 - \bar{a}}.$$

5. (Exercise 38)

(a) The LP formulation is:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & p^{(i)T} x \geq \lambda \text{ for all } i \\ & \sum_{i=1}^{\lfloor n/20 \rfloor} x_{[i]} \leq 0.9 \\ & 1^T x \leq 1 \\ & x \geq 0 \end{array}$$

(b) The LP formulation is:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & (\bar{p} + By)^T x \geq \lambda \\ & \sum_{i=1}^{\lfloor n/20 \rfloor} x_{[i]} \leq 0.9 \\ & 1^T x \leq 1 \\ & y_i \leq l \text{ for all } i = 1, \ldots, m \\ & -y_i \leq l \text{ for all } i = 1, \ldots, m \\ & l \leq 1 \\ & x \geq 0 \end{array}$$

(c) The LP formulation is:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & (\bar{p} + y)^T x \geq \lambda \\ & \sum_{i=1}^{\lfloor n/20 \rfloor} x_{[i]} \leq 0.9 \\ & By \leq d \\ & 1^T x \leq 1 \\ & x \geq 0 \end{array}$$

6. (Exercise 45)

The initial basis matrix is

$$B = \begin{bmatrix} 12 & 4 & 1 \\ 6 & -7 & 18 \\ 1 & 17 & 3 \end{bmatrix}.$$

We have $B^{-1}b = (1, 0, 0)$, so the corresponding extreme point is $(1, 0, 0, 0, 0)$. The reduced costs of non-basic variables are

$$\bar{c}_4 = c_4 - [c_1, c_2, c_3]\, B^{-1} A_4 = -464$$

and

$$\bar{c}_5 = c_5 - [c_1, c_2, c_3]\, B^{-1} A_5 = -5.3.$$

Applying Bland's rule, we decide x_4 will enter the basis.
Next we determine the vector d:

$$(d_1, d_2, d_3) = -B^{-1} A_4 = (2, -1, -1).$$

There are two negative components, so we have to compare the ratios

$$-x_2/d_2 = 0, \;\; -x_3/d_3 = 0.$$

Using Bland's rule, we select x_2 to leave the basis.
The new basis is $\{1, 3, 4\}$. The basis matrix is

$$B = \begin{bmatrix} 12 & 1 & -19 \\ 6 & 18 & -1 \\ 1 & 3 & 18 \end{bmatrix},$$

and the corresponding extreme point is

$$x = (1, 0, 0, 0, 0).$$

The reduced costs are

$$\bar{c}_2 = c_2 - [c_1, c_3, c_4]\, B^{-1} A_2 = 464$$

and

$$\bar{c}_5 = c_5 - [c_1, c_3, c_4]\, B^{-1} A_5 = -1.1,$$

so x_5 will enter the basis.
Next we determine the vector d:

$$(d_1, d_3, d_4) = -B^{-1} A_5 = (-0.68, 0.95, -0.01).$$

There are two negative components, so we have to compare the ratios

$$-x_1/d_1 = 1.48, \;\; -x_4/d_4 = 0.$$

The smallest ratio is $-x_4/d_4$, so x_4 will leave the basis.
The new basis is $\{1, 3, 5\}$. The basis matrix is

$$B = \begin{bmatrix} 12 & 1 & 7 \\ 6 & 18 & -13 \\ 1 & 3 & -2 \end{bmatrix},$$

and the corresponding extreme point is

$$x = (1, 0, 0, 0, 0).$$

The reduced costs are

$$\bar{c}_2 = c_2 - [c_1, c_3, c_5]\, B^{-1} A_2 = 582$$

and

$$\bar{c}_4 = c_4 - [c_1, c_3, c_5]\, B^{-1} A_4 = 118.$$

Both are nonnegative, so x is optimal.
We can recover the dual optimal solution as

$$y = B^{=T}(c_1, c_3, c_5) = (-1.8286, 1.3143, -9.9429).$$