IE 513 Linear Programming
Fall 2000
Homework 4

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1. Show directly that if the primal problem $\min cx$ subject to $Ax = b$ and $x \geq 0$ has no feasible solutions, and if the dual problem has a feasible solution, then the dual problem is unbounded. (Hint: Use Farkas' Lemma, If the system $Ax \geq b$ and $x \geq 0$ has no solutions, then the system $wA \leq 0$, $w \geq 0$, $wb > 0$ has a solution.)
2. Use the main duality theorem to prove Farkas' lemma. (Hint: Consider the following pair of primal and dual linear programs: $\min \mathbf{0}x$ subject to $Ax = b$, $x \geq 0$ and $\max wb$ subject to $wA \leq 0$, w unrestricted.
3. Two players are involved in a competitive game. One player, called the row player, has two strategies available; the other player, called the column player, has three strategies available. If the row player selects strategy i and the column player selects strategy j the payoff to the row player is c_{ij} and the payoff to the column player is $-c_{ij}$. Thus, the column player loses what the row player wins and vice versa – a *a two-person zero-sum game*. The following are the values of the payoff table: $c_{11} = 2$, $c_{12} = -1$, $c_{13} = 0$, $c_{21} = -3$, $c_{22} = 1$, $c_{23} = 1$. Let x_1, x_2, x_3 be the probabilities with which the column player selects the various strategies over many plays of the game. Thus, $x_1 + x_2 + x_3 = 1$, $x_1, x_2, x_3 \geq 0$. If the column player applies these probabilities to the selection of his strategy for any play of the game, consider the row player's options. If the row player selects row 1, then his expected payoff is $2x_1 - x_2$. If the row player selects row 2, his payoff is $-3x_1 + x_2 + x_3$. Wishing to minimize the maximum expected payoff to the row player, what is the problem that the column player should solve? Give the dual of this problem. Interpret the dual problem (Hint: consider the row player's problem). Solve the dual problem graphically. Use the dual solution to compute the column player's probabilities. Interpret the complementary slackness conditions for this two-person zero-sum game.
4. Solve the following problem by the primal-dual algorithm

$$\min x_1 + 2x_3 - x_4$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 6$$

$$2x_1 - x_2 + 3x_3 - 2x_4 \geq 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

5. Suppose that at the end of the restricted primal problem (in the context of the primal-dual method) we have $x_0 > 0$ and $v^* a_j \geq 0$ for all j. Using Farkas' lemma, show that the primal problem has no feasible solution.
6. A company has requirements for a certain type of machine during the next N months of D_i per month, $i = 1, 2, \ldots, N$. Those requirements must be met, although excesses are permitted to any desirable extent. No machines are available at the beginning of month 1. At the beginning of of each of the the N months, the company may purchase machines. The machines are delivered immediately; that is, there is no lead time. The company may purchase machines that last one month, two months, and so on up to M months. The number of monthsa machine is usable is called its service life. The cost of a machine depends on the month in which it is bought and on its service life. In particular, a machine bought in the ith month with service life of k months costs c_{ik} dollars. Naturally, a machine that lasts longer costs more, so that $c_{ip} < c_{is}$ for $p < s$. Formulate

a mathematical statement for this problem, assuming that the objective is to minimize the sum of the monthly costs. Let x_{ik} be the number of machines bought in month i with service like of k months. Formulate the *general* problem as a flow problem, use $N = 4$, and $M = 3$ to illustrate, summarizing your work on a network. (Hint: Consider elementary row operations to obtain a 1 and -1 in every column.)