IE513 Homework 4 Solutions

1. We can write the primal problem as

$$\begin{array}{ll} \text{minimize} & cx \\ \text{subject to} & Ax \leq b \\ & -Ax \leq -b \\ & x \geq 0 \end{array}$$

then the dual LP is

$$\begin{array}{ll} \text{maximize} & wb^* \\ \text{subject to} & wA^* \leq c \\ & w \geq 0 \end{array}$$

where $A = \begin{bmatrix} A \\ -A \end{bmatrix}$ and $b = \begin{bmatrix} b \\ -b \end{bmatrix}$. Since, primal problem has no feasible solutions $A^*x \leq b$ and $x \geq 0$ is infeasible. Then, by Farkas' Lemma, $w \geq 0,\ \ wA^* \leq 0,\ \ wb^* > 0$ has a solution. We also know that our dual problem has a feasible solution, i.e., there exists a w^* such that $w^*A^* \leq c$. Combining both results gives the equation

$$(w^* + w)A^* \leq c.$$

Now, for any $\alpha \geq 0$

$$(w^* + \alpha w)A^* \leq c$$

is also feasible and the objective value of the dual problem is

$$(w^* + \alpha w)b^* = w^*b^* + \alpha wb^*.$$

Since $w^*b^* > 0$ and $\alpha \geq 0$, as $\alpha \to \infty$ the objective value also goes to infinity. Hence, the dual problem is unbounded.

2. Let's write the Farkas' Lemma for the given primal dual pair:
i) there exists an $x \geq 0$ such that $Ax = b$.
ii) there exists a w such that $wA \leq 0$ and $wb < 0$.
Assume i) is true. Then there exists an $x \geq 0$ such that $Ax = b$. This means that the primal problem has a feasible solution with an objective value 0. Then by strong duality theorem, the dual problem has an objective value 0, i.e., $wb = 0$ which implies $wb < 0$ is false.
Assume ii) is true. Then, $wA \leq 0$ and $wb < 0$ which means dual problem is feasible with an objective value strictly less than zero. Again, by applying strong duality theorem, the primal problem should have objective value less than zero. However, the primal problem can not have negative optimal value. So, i) is false.

3. The column player should solve the problem

$$\begin{array}{ll} \text{minimize} & \max\,(2x_1 - x_2,\ \ -3x_1 + x_2 + x_3) \\ \text{subject to} & x_1 + x_2 + x_3 = 1 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{array}$$

To linearize the above problem, introduce a variable y:

$$\begin{array}{ll} \text{minimize} & y \\ \text{subject to} & 2x_1 - x_2 \leq y \\ & -3x_1 + x_2 + x_3 \leq y \\ & x_1 + x_2 + x_3 = 1 \\ & x_1,\ x_2,\ x_3 \geq 0 \\ & y \text{ unrestricted.} \end{array}$$

Dual of the above problem if we introduce w_1 and w_2 for the first two constraints and z for the third constraint:

$$\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & 2w_1 - 3w_2 + z \leq 0 \\ & -w_1 + w_2 + z \leq 0 \\ & w_2 + z \leq 0 \\ & -w_1 - w_2 = 1 \\ & w_1,\ w_2, \leq 0 \\ & z \text{ unrestricted.} \end{array}$$

If we substitute the variables $v_1 = -w_1$ and $v_2 = -w_2$ then the problem becomes

$$\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & 2v_1 - 3v_2 \geq z \\ & -v_1 + v_2 \geq z \\ & w_2 \geq z \\ & v_1 + v_2 = 1 \\ & v_1,\ v_2, \geq 0 \\ & z \text{ unrestricted.} \end{array}$$

If we write this problem similar to the above, then our problem becomes

$$\begin{array}{ll} \text{maximize} & \min\,(2v_1 - 3v_2,\ \ -v_1 + v_2,\ \ v_2) \\ \text{subject to} & v_1 + v_2 = 1 \\ & v_1,\ v_2, \geq 0 \end{array}$$

This is the problem that the row player should solve in order to maximize the minimum expected payoff. v_1 and v_2 are the respective probabilities which the row player selects the various strategies.

After solving the dual problem graphically, optimal solution is $(4/7,\ 3/7,\ -1/7)$ for $v_1,\ v_2,$ and z respectively. From complementary slackness conditions, $x = (2/7,\ 5/7,\ 0)$.

At the optimality $x_3 = 0$. This is expected since, if the column player selects column 3, the expected gain is $-x_3$ which is always negative for $x_3 > 0$. So, the column player should never select column 3. Similar interpretations can be made. The complementary slackness conditions are $(2x_1 - x_2 - y)v_1 = 0,\ \ (-3x_1 + x_2 + x_3 - y)v_2 = 0$ for primal problem and $(2v_1 - 3v_2 - z)x_1 = 0,\ \ (-v_1 + v_2 - z)x_2 = 0,\ \ (-v_2 - z)x_3 = 0$ for dual problem.

4. The standard form of the problem is

$$\begin{array}{ll} \text{minimize} & x_1 + 2x_3 - x_4 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 + x_5 = 6 \\ & 2x_1 - x_2 + 3x_3 - 2x_4 - x_6 = 5 \\ & x_1, \ldots x_6 \geq 0. \end{array}$$

Then the dual problem is

$$\begin{array}{ll} \text{maximize} & 6w_1 + 5w_2 \\ \text{subject to} & w_1 + 2w_2 \leq 1 \\ & w_1 - w_2 \leq 0 \\ & w_1 + 3w_2 \leq 2 \\ & w_1 - 2w_2 \leq -1 \\ & w_1 \leq 0 \\ & -w_2 \leq 0 \\ & w_1,\ \ w_2 \ \text{unrestricted.} \end{array}$$

Choose $w^T = (-1,\, 0)$ which is feasible for dual LP. Then $Q = \{4,\, 6\}$ where Q is the index set whose elements are indices of tight constraints of the original dual problem. Then the restricted primal problem is:

$$\begin{array}{ll} \text{minimize} & x_7 + x_8 \\ \text{subject to} & x_4 + x_7 = 6 \\ & -2x_4 - x_6 + x_8 = 5 \\ & x_1, \ldots x_8 \geq 0. \end{array}$$

Optimal solution of this problem is $(x_4,\, x_6,\, x_7,\, x_8) = (0,\, 0,\, 6,\, 5)$. Dual of the primal restricted problem is:

$$\begin{array}{ll} \text{maximize} & 6v_1 + 5v_2 \\ \text{subject to} & v_1 - 2v_2 \leq 0 \\ & -v_2 \leq 0 \\ & v_1 \leq 1 \\ & v_2 \leq 1 \\ & v_1,\ \ v_2 \ \text{unrestricted.} \end{array}$$

Last two constraints of the primal problem is tight from the complementary slackness conditions, so $v^* = (1, 1)$. After computing the $v^* a_j$, $v^* a_1 = 3, v^* a_2 1 = 0, v^* a_3 = 4, v^* a_4 = -1, v^* a_5 = 1, v^* a_6 = -1$ and computing

$$Q = \min(-(-1-1)/3,\, -(-1-2)/4,\, -(-1-0)/1) = 2/3$$

implies x_1 enters and $w = (-1, 0) + 2/3(1, 1) = (-1/3, 2/3)$ which is new dual feasible solution. Then new primal restricted problem is

$$\begin{array}{ll} \text{minimize} & x_7 + x_8 \\ \text{subject to} & x_1 + x_7 = 6 \\ & 2x_1 + x_8 = 5 \\ & x_1, \ldots x_8 \geq 0. \end{array}$$

The optimal point to this problem is $(x_1,\, x_7,\, x_8) = (2.5,\, 3.5,\, 0)$. The dual of the restricted problem is:

$$\begin{array}{ll} \text{maximize} & 6v_1 + 5v_2 \\ \text{subject to} & v_1 + 2v_2 \leq 0 \\ & v_1 \leq 1 \\ & v_2 \leq 1 \\ & v_1,\ \ v_2 \ \text{unrestricted.} \end{array}$$

First two constraints of the primal problem is tight from the complementary slackness conditions, so $v^* = (1, -1/2)$. Then computing $Q = 1/3$ implies x_4 and x_5 enters the problem and new $w = (0, 1/2)$. New $Q = \{1, 4, 5\}$ and the new primal restricted problem is

$$\begin{array}{ll} \text{minimize} & x_7 + x_8 \\ \text{subject to} & x_1 + x_4 + x - 5 + x_7 = 6 \\ & 2x_1 - 2x - 4 - x_6 - x_8 = 5 \\ & x_1, \ldots x_8 \geq 0. \end{array}$$

Optimal solution of this problem is $(x_1,\, x_4,\, x_5,\, x_6,\, x_7,\, x_8) = (2.5,\, 0,\, 3.5,\, 0,\, 0,\, 0)$ with optimal value 0. So, we achieved the optimal solution to the original primal problem $(x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6) = (2.5,\, 0,\, 0,\, 0,\, 3.5,\, 0)$ and the dual optimal solution is $(w_1,\, w_2) = (0,\, 1/2)$. Both problems have optimal value 5/2.

5. The restricted primal problem

$$\begin{array}{ll} \text{minimize} & \sum_{j \in Q} 0x_j + 1x_a \\ \text{subject to} & \sum_{j \in Q} a_j x_j + 1x_a = b \\ & x_j \geq 0 \ \ \forall j \\ & x_a \geq 0. \end{array}$$

The dual of this problem if we substitute $z = -v$ and write it as a minimization problem, then our problem becomes

$$\begin{array}{ll} \text{minimize} & zb \\ \text{subject to} & za_j \geq 0 \ \ \forall j \in Q \\ & z \geq -1 \\ & z \ \text{unrestricted.} \end{array}$$

It is given in the problem that $z^*b < 0$ and $z^*a_j \geq 0$ for all j. Since our aim is to show that primal is infeasible, we can take $c = 0$ in the original problem. In this way the dual problem can be formulated as follows:

$$\begin{array}{ll} \text{maximize} & wb \\ \text{subject to} & wA \leq 0 \\ & w \ \text{unrestricted.} \end{array}$$

If we change the problem $y = -w$, the problem is equivalent to

$$\begin{array}{ll} \text{minimize} & yb \\ \text{subject to} & yA \geq 0 \\ & y \text{ unrestricted.} \end{array}$$

It is given that this problem is feasible. Then by Farkas' Lemma, there exist no x such that $Ax = b$ and $x \geq 0$. So, the primal problem has no feasible solution.

6. The LP formulation of the problem assuming $M = N$ is

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \sum_{k=1}^{M} c_{ik} x_{ik} \\ \text{subject to} & x_{11} + x_{12} + \ldots + x_{1M} = D_1 \\ & x_{12} + \ldots + x_{1M} + x_{21} + \ldots + x_{2M} = D_2 \\ & \vdots \\ & x_{1M} + x_{2(M-1)} + x_{2M} + \ldots + x_{N1} + \ldots + x_{NM} = D_N \end{array}$$

For $N = 4$ and $M = 3$, the problem is

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{4} \sum_{k=1}^{3} c_{ik} x_{ik} \\ \text{subject to} & x_{11} + x_{12} + x_{13} = D_1 \\ & x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = D_2 \\ & x_{13} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} = D_3 \\ & x_{23} + x_{32} + x_{33} + x_{41} + x_{42} + x_{43} = D_4 \end{array}$$