IE 515 Convex Analysis
Spring 2007
Final Examination, Take-home, May 18, 2007 Friday
Instructor: M.Ç. Pınar

There are 7 questions, with total weights equal to 30 points. Please read the questions carefully. Write legibly and show all your work.

The exam is to be solely based on your individual work. You are not allowed to consult anyone else while solving the questions, nor discuss the exam among yourselves. If you happen to come across answers to any of the exam questions in any source, printed or electronic, fully or partially, you should indicate this in your answer.

You should return the exam to myself on Monday, May 21, 2007 by 10 am.

Please sign this cover page on your exam sheet and staple it to the front of your answers:

> **I hereby solemnly declare that the exam represents entirely my own work, and that I have not sought help from anyone, nor have I discussed the contents of the exam with anyone else.**

NAME:

SIGNATURE:

1.[3 points] Let A, C and D be given matrices with A non-vacuous (non-zero). Then by a well-known theorem of the alternative, either

$$\text{(I) } Ax > 0 \; Cx \geq 0 \; Dx = 0$$

has a solution x or

$$\text{(II) } A^T y_1 + C^T y_3 + D^T y_4 = 0, y_1 \geq 0, \; y_3 \geq 0$$

has a solution y_1, y_3, y_4 but never both.

(a) Prove that (I) implies the negation of (II). *Hint*: Assume both hold, and try to get a contradiction.

(b) Using the above theorem of the alternative, can you prove the following: *For each given matrix A, either $Ax > 0$ has a solution x or $A^T y = 0, y \geq 0$ has a solution but never both.*

2.[4 points] We define the *monotone non-negative cone* as

$$K_{m+} = \{x \in \mathbb{R}^n | x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}.$$

Find the dual cone to K_{m+}. *Hint* : Use the identity

$$\begin{aligned} \sum_{i=1}^{n} x_i y_i &= (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \ldots \\ &+ (x_{n-1} - x_n)(y_1 + \ldots + y_{n-1}) + x_n(y_1 + \ldots + y_n). \end{aligned}$$

3. [3 points] Let $p, q \in \mathbb{R}^n$ represent two probability distributions on $\{1, \ldots, n\}$ (so, $p, q \geq 0$, $\sum_{i=1}^n p_i = 1, \sum_{i=1}^n q_i = 1$). The maximum probability distance $d_{mp}(p, q)$ between p and q is defined as the maximum difference in probability assigned by p and q, over all events:

$$d_{mp}(p, q) = \max\{|\text{prob}(p, C) - \text{prob}(q, C)| : C \subseteq \{1, \ldots, n\}\}.$$

Here $\text{prob}(p, C)$ is the probability of C under p. Find a simple expression for d_{mp}, involving $\|p - q\|_1$, and show that d_{mp} is a convex function on $\mathbb{R}^n \times \mathbb{R}^n$.

4.[5 points] Let $C \in \mathbb{R}^n$ be a closed, convex set and $f : C \mapsto \mathbb{R}$ a function with continuous partial derivatives $\partial f / \partial x_i, i = 1, \ldots, n$. The first order necessary condition for a local minimizer x^* of f on C is

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C. \tag{1}$$

(a) For the problem of $\min\{f(x)|Ax = b\}$ where $f : \mathbb{R}^n \mapsto \mathbb{R}$, A is $m \times n$ matrix and $b \in \mathbb{R}^m$, find a simple expression for the necessary condition (1) for a local minimizer.

(b) Apply the result of part (a) to the problem

$$\max\{x_1^{\alpha_1} \dots x_n^{\alpha_n} : x_1 + \dots + x_n = 1, x_i \geq 0, i = 1, \dots, n\}$$

to compute a closed-form expression for the optimal solution. *Hint*: Observe that the problem above is equivalently posed as

$$\min\{-\alpha_1 \log x_1 + \dots + (-\alpha_n) \log x_n : x_1 + \dots + x_n = 1, x_i > 0, i = 1, \dots, n\},$$

and that positivity conditions can be dropped. Argue why this is the case and proceed. log denotes natural logarithm.

5. [5 points] Derive a dual problem using Lagrange duality for

$$\min_x \sum_{i=1}^{N} \|A_i x + b_i\|_2 + (1/2)\|x - x_0\|_2^2$$

where the problem data are $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$ and $x_0 \in \mathbb{R}^n$. First introduce new variables $y_i \in \mathbb{R}^{m_i}$ and create constraints $y_i = A_i x + b_i$. *Hint*: You have to use Cauchy-Schwarz inequality.

6. [6 points] Consider the following function

$$f(x_1, x_2) = \begin{cases} (x_1 - x_2)(\log(x_1) - \log(x_2)) & \text{if } x_1 > 0, x_2 > 0 \\ 0 & \text{if } x_1 = x_2 = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

(a) Show that f is a convex function.

(b) Is f l.s.c. at $(0, 0)$?

(c) Find the subdifferential $\partial f(0, 0)$ at the origin.

7. [4 points] Consider the convex minimization problem

$$\min\{\frac{\|x\|_2^2}{2} : \langle a_i, x \rangle \leq b_i, i = 1, \dots, m\}$$

where $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}$ are given. Assume that the linear inequality system $\langle a_i, x \rangle \leq b_i, i = 1, \dots, m$ has a solution.

(a) Give a geometric interpretation of the problem, and show that f defined as

$$f(x) = \begin{cases} \frac{\|x\|_2^2}{2} & \text{if } Ax \leq b \\ +\infty & \text{otherwise} \end{cases}$$

is a proper, convex function.

(b) Consider the perturbation function

$$\Phi(x) = \begin{cases} \frac{\|x\|_2^2}{2} & \text{if } Ax \leq b + p \\ +\infty & \text{otherwise} \end{cases}$$

Calculate the dual problem with respect to Φ.

(c) Give a description of the value function $v(p)$.

(d) Under what (additional) conditions can be concluded that strong duality holds, that is, $\inf(P) = \sup(D)$?