Name:

MATH 520 Linear Algebra
Fall 2002
Examination III
18.12.2002

There are 7 questions and the weights add up to 20. Prove or disprove means: either you choose to give a proof or you give a counterexample. Please answer the questions in the space provided.

1.[1 point] Prove or disprove the following statement:

> If V is a real inner product space, then for arbitrary non-zero vectors v and w in V, we have
> $$-1 \leq \frac{\langle u, v\rangle}{\|u\|\|v\|} \leq 1.$$

2.[3 points] Find the closest vector to $(2, 1, 2, -1, -2)$ in the subspace spanned by

$$(1, 1, 0, 0, -1), (1, -1, 1, 0, 0), (0, 0, 0, 1, 0).$$

3. [3 points]
Prove or disprove the following statement:

> If $\{v_1, \ldots, v_n\}$ is an **orthogonal** set of vectors of an inner product space V, then for each $v \in V$ we have
>
> $$\|v\| \geq \sqrt{\sum_{k=1}^{n} \frac{\langle v, v_k \rangle^2}{\langle v_k, v_k \rangle}}.$$
>
> Equality holds iff
>
> $$v = \sum_{k=1}^{n} \frac{\langle v, v_k \rangle^2}{\langle v_k, v_k \rangle} v_k.$$

4. [2 points] Find the orthogonal projection of the function e^x on the subspace of $C[-\pi, \pi]$ (with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\bar{g}(x)dx$) generated by $\cos x$ and $\sin x$. (Recall that $\sinh x = \frac{e^x - e^{-x}}{2}$).

5.[4 points] We are given the vector space of continuous functions on $[-1,1]$, the space $C[-1,1]$ with inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)\bar{g}(x)dx$. Let $V_+ = \{f \in C[-1,1] : f(x) = f(-x)\}$ be the subspace of even functions and $V_- = \{f \in C[-1,1] : f(x) = -f(-x)\}$ be the subspace of odd functions. Prove or disprove:

$$V = V_+ \oplus V_-.$$

6.[4 points] Consider a **complex** vector space V of dimension n. You are given two operators F and G on V such that

$$GF - FG = \alpha F$$

where α is a complex nonzero number. Let v be an eigenvector of G corresponding to eigenvalue λ.

(a) Show (using induction argument) that for all integer k ($k = 1, \ldots$) we have

$$GF^k(v) = (\lambda + \alpha k)F^k(v)$$

(b) Deduce from (a) the existence of an integer $k \in \{0, 1, \ldots, n\}$ such that $F^k(v) = 0$ and that F is not injective.

7.[3 points] We are given n real nonzero numbers $b_1, \ldots, b_n$ and the $n \times n$ matrix A where the (i, j) entry a_{ij} is given by

$$a_{ij} = b_i b_j.$$

Therefore, A is the matrix representation of an operator T of a real n-dimensional vector space V in a given basis $\{e_1, \ldots, e_n\}$.

Using the definition of an eigenvalue and eigenvector, show that $\lambda_1 = 0$ is an eigenvalue of T and that the eigenspace E_1 associated with λ_1 is a subspace of dimension $n - 1$ that you should determine explicitly.