Extensions of Beck-Teboulle Global Optimality Conditions for Bivalent Quadratic Optimization

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- Beck and Teboulle SIAM J. Opt. (2000) (henceforth abbreviated **BT**) considered the **unconstrained bivalent** quadratic optimization problem

  [**P**]

$$\begin{array}{ll} \min & \frac{1}{2}x^T Q x + c^T x \\ \text{s.t.} & x \in \{-1, 1\}^n \end{array}$$

  where $Q \in \mathcal{S}^n$, (a symmetric $n \times n$ matrix), and $c \in \mathbb{R}^n$.

- They gave necessary and sufficient (different) conditions for global optimality in **P**.

- Example: MAX-CUT

- We consider the bivalent quadratic optimization problem

[**QP**]

$$
\begin{aligned}
\min \quad & \frac{1}{2}x^T Q x + c^T x \\
\text{s.t.} \quad & x^T E_i x + d_i^T x = f_i, \forall\ i = 1, \ldots, m \\
& x \in \{-1, 1\}^n
\end{aligned}
$$

where $Q \in \mathcal{S}^n$, and $E_i \in \mathcal{S}^n$, $\forall i = 1, \ldots, m$, $c, d_i \in \mathbb{R}^n$, and $f_i \in \mathbb{R}$ for all $i = 1, \ldots, m$.

- The purpose of this talk is to present a sufficient condition for global optimality in **QP**.

- A straightforward extension of **BT**.

- Examples: Graph Maximum Bisection, $s - u - v$ MAX-CUT, $s - t$ MAX CUT ($c \equiv 0$ in all examples).

BT Sufficient Condition for Global Optimality in P

- Let $x \in D = \{x : x \in \{-1, 1\}^n\}$, and $X = \mathrm{Diag}(x)$, where Diag creates a diagonal matrix consisting of the elements of x.

- Then, $x = Xe$ where e is the n vector of ones.

- If

$$\lambda_{\min}(Q)e \geq XQXe + Xc$$

  then x is a global optimal solution for **P**.

- **Proof**. Based on Lagrange Duality.

- Condition is very easy to check!

- **Necessary Condition** is simpler:

$$\mathrm{diag}(Q) \geq XQXe + Xc$$

  where $\mathrm{diag}(Q) = (q_{11}, q_{22}, \ldots, q_{nn})^T$.

Sufficient Condition for Global Optimality in QP

- Let D^T denote the $n \times m$ matrix with columns d_i, $i = 1, \ldots, m$.

- Given feasible x, if there exists $z \in \mathbb{R}^m$ which solves

  $$Q + \mathrm{Diag}(-XQx - X(\sum_{i=1}^{m} z_i E_i)x - Xc - \frac{1}{2} X D^T z) + \sum_{i=1}^{m} z_i E_i \succeq 0$$

  then x is a global optimal solution for **QP**.

- **Proof.** Direct extension of **BT**. Assuming condition holds, construct dual vector y (belonging to domain of the dual function) using x and z such that duality gap is zero. The trick is to exhibit such a y using the $\{-1, 1\}$ bivalency of x.

- Sufficient Condition involves the solution of a Positive Semidefinite Feasibility Problem or Linear Matrix Inequality (LMI).

Only Linear Constraints in QP

- Consider the linearly constrained bivalent QP

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Q x + c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in \{-1, 1\}^n \end{aligned}$$

- Sufficient condition simplifies and reduces to
- Given feasible x, if there exists $z \in \mathbb{R}^m$ satisfying

$$\lambda_{\min}(Q)e \geq XQXe + Xc + XA^T z$$

  then x is global optimal.

- A linear inequality system in z
- Reduces to **BT** s.c. if linear constraints are absent.

Application: Quadratic Knapsack

- Consider

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Q x + c^T x \\ \text{s.t.} \quad & a^T x \leq b \\ & x \in \{-1, 1\}^n \end{aligned}$$

- after transformation from $\{0, 1\}$ to $\{-1, 1\}$ and from max to min.

- Given feasible x such that $a^T x = b$, if there exists $z \in \mathbb{R}_+$ satisfying

$$\lambda_{\min}(Q)e \geq XQXe + Xc + zXd$$

then x is a global optimal solution for **QP**.

Linear Objective Function in QP

- Consider

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & x^T E_i x + d_i^T x = f_i, \forall\, i = 1, \ldots, m \\
& x \in \{-1, 1\}^n
\end{aligned}
$$

- Given feasible x, if there exists $z \in \mathbb{R}^m$ which solves

$$\mathrm{Diag}(-\frac{1}{2} X D^T z - X(\sum_{i=1}^m z_i E_i)x - Xc) + \sum_{i=1}^m z_i E_i \succeq 0$$

then x is a global optimal solution for **QP**.

Linear Bivalent Optimization

- Consider the linearly constrained bivalent QP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in \{-1, 1\}^n \end{aligned}$$

- Sufficient condition reduces to
- Given feasible x, if there exists $z \in \mathbb{R}^m$ satisfying

$$Xc + XA^T z \le 0$$

then x is global optimal.

- A linear inequality system in z

Convex Case

- **BT** considered the case where $Q \succeq 0$ and the convex relaxation:

  [**CP**]

$$
\begin{array}{ll}
\min & \frac{1}{2}x^T Q x + c^T x \\
\text{s.t.} & x_i^2 \leq 1, \forall\, i = 1, \ldots, n
\end{array}
$$

- Necessary and sufficient condition for x to be optimal in both **P** and **CP**:

$$XQXe + Xc \leq 0$$

- **Proof**. Similar to previous proofs. Use Lagrange Duality.

Convexity in Linearly Constrained QP

- Assume $Q \succeq 0$ and consider the convex relaxation:

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Qx + c^T x \\ \text{s.t.} \quad & Ax = b \\ & x_i^2 \leq 1, \forall\, i = 1, \ldots, n \end{aligned}$$

- Necessary and sufficient condition for x to be optimal in both linearly constrained **QP** and its convex relaxation:

- $\exists z \in \mathbb{R}^m$ such that

$$XQXe + Xc + XA^T z \leq 0$$

Convexity in QP

- Suppose $Q \succeq 0$ and $E_i \succeq 0$, $i = 1, \ldots, m$ and consider the convex relaxation:

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Q x + c^T x \\ \text{s.t.} \quad & x^T E_i x + d_i^T x \le f_i, \forall\, i = 1, \ldots, m \\ & x_i^2 \le 1, \forall\, i = 1, \ldots, n \end{aligned}$$

- Assume Slater condition holds.
- Necessary and sufficient condition for x to be optimal in **QP** and its convex relaxation:
- $\exists z \in \mathbb{R}^m$ such that

$$XQXe + X(\sum_{i=1}^m z_i E_i)x + Xc + \frac{1}{2}XD^T z \le 0$$

A Matrix Extension to BT

- Let us consider the quadratic program in matrix variables X:

$$\begin{aligned} \min \quad & \mathrm{Tr} AXBX^T + \mathrm{Tr} CX^T \\ \text{s.t.} \quad & XX^T = I \end{aligned}$$

  where A, B are symmetric $n \times n$ matrices, C and X are an $n \times n$ matrices.

- Nonconvex quadratic program defined over orthonormal matrices (Stiefel manifold).

- Well-known relaxation for the Quadratic Assignment Problem: the **Eigenvalue Bounds** (for $C \equiv 0$).

- **Sufficient Condition** for optimality of feasible (i.e., orthonormal) X:

$$\lambda_{\min}(B \otimes A) \geq \lambda_{\max}(AXBX^T + CX^T)$$

  where $\otimes$ denotes Kronecker product.

- **Proof.** Straightforward matrix extension of BT.

A Matrix Extension to BT II

- Let us consider again the quadratic program in matrix variables X: with row and column sum constraints:

$$\begin{aligned} \min \quad & \mathrm{Tr} AXBX^T + \mathrm{Tr} CX^T \\ \text{s.t.} \quad & XX^T = I \\ & Xe = e \\ & X^T e = e \end{aligned}$$

- Nonconvex quadratic program defined over orthonormal matrices with unit rown and column sums.

- Tighter relaxation for the Quadratic Assignment Problem

- **Sufficient Condition** for optimality of feasible X: Exist $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ satisfying the LMI

$$B \otimes A - I \otimes \left(AXBX^T + CX^T - (ue^T + ew^T)X^T \right) \succeq 0$$

Quadratic Assignment Problem

- Let us consider the QAP:

$$\begin{aligned} \min\ \ & \mathrm{Tr} AXBX^T + \mathrm{Tr} CX^T \\ \text{s.t.}\ \ & XX^T = I \\ & Xe = e \\ & X^T e = e \\ & X \geq 0 \end{aligned}$$

- **Sufficient Condition** for optimality of feasible X: Exist $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ and $T \in \mathbb{R}_+^{n \times n}$ with $T_{ij} = 0$ for all (i,j) such that $X_{ij} > 0$ satisfying the LMI

$$B \otimes A - I \otimes \left(AXBX^T + CX^T - (ue^T + ew^T + T)X^T \right) \succeq 0$$