On Closed-Form Solutions of a Resource Allocation Problem in Parallel Funding of R & D Projects

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Abstract

In order to reduce the risk of complete failure, research managers often adopt a parallel strategy by simultaneously funding several R & D activities and several research teams within each activity. The parallel strategy requires the allocation of an available budget to a number of R & D activities, the determination of the number of research teams within each activity and the amount of funding they receive. We consider a formulation of this problem as a nonlinear resource allocation problem by Gerchak and Kilgour (1999), and present a sufficient condition as a function of problem parameters, under which closed-form solutions to the problem are obtained.

Key words: Project Selection; Parallel Strategy, Nonlinear Resource Allocation.

1 Introduction

The purpose of this paper is to investigate closed-form solutions to a nonlinear resource allocation problem arising in the selection process among competing research and development (R & D) activities. (R&D) project managers face the challenge of exploring several choices to attain a particular objective. However, the outcome of these choices is usually uncertain, making the selection process a difficult one. Abernathy and Rosenbloom [1] suggested the use of a parallel strategy as a tool to deal with this uncertainty in a successful way. They define a parallel strategy as "the simultaneous pursuit of two or more distinct approaches to a single task, when successful completion of any one would satisfy the task requirements." However, the adoption of the parallel strategy in a R&D project is associated with the crucial strategic question of determining how many parallel teams or approaches to a particular objective to fund. Gerchak and Kilgour [2] modeled different objectives to determine how many parallel independent research teams of equal potential to fund within only one research and development activity. Gerchak [3] extended these ideas to the case where more than a single activity is involved, and these activities have different priorities reflected in different weights. In another study by Gerchak [5], a model with a single activity is considered where the achievements of competing teams are allowed to be interdependent. Gerchak and Parlar [4] tackle a similar problem using game theoretic approaches. In the present paper, we identify conditions under which the nonlinear and nonconvex optimization problem introduced by Gerchak and Kilgour [2] admits closed-form solutions. For a general and detailed exposition of resource allocation problems the interested reader is directed to the book [6].

2 The Optimization Problem and Main Results

Gerchak and Kilgour [2] modeled the resource allocation problem as follows. They assume that (1) the future achievement of a research team is a continuous nonnegative random variable, (2) the research teams are independent, (3) the achievements of the research teams have identical probability distributions, given that their funding is done on an equal basis, and (4) the achievements of the funded teams are exponentially distributed. Suppose that the total available budget is B, and there are M potential research activities. For $j = 1, 2, .., M$, let B_j be the budget allocated to activity j and n_j be the number of parallel research teams to work on activity j, each of which receive equal funding which amounts to B_j/n_j. The achievement of each team within activity j has an identical distribution function given by $F_j(x; B_j/n_j)$, where B_j/n_j is related to the parameter of the distribution. The optimization problem consists in maximizing a weighted sum of the probabilities that the most successful team within each activity exceeds a specific threshold value. Since the probability that the best team in activity j exceeds the threshold T_j can be written as $1-[F_j(T_j; B_j/n_j)]^{n_j}$, the optimization problem is formulated as follows (see [2]):

$$\max_{B_1,\ldots,B_M,n_1,\ldots,n_M} \sum_{j=1}^{M} p_j\{1 - [F_j(T_j; B_j/n_j)]^{n_j}\},$$

subject to

$$\sum_{j=1}^{M} B_j = B,$$

where p_j denotes the positive weight attached to activity j.

Gerchak and Kilgour [2] sets the parameter λ of the exponential distribution to $\lambda = L(n/B)^{\alpha}$ where $\alpha \in (0, 1]$ reflects the sensitivity of a research team to the budget allocation, and L is an appropriate constant. The case $\alpha = 1$ indicates a high sensitivity where larger budget allocations would induce higher achievements. With multiple activities this problem translates into:

$$\max_{B_1,\ldots,B_M,n_1,\ldots,n_M} \sum_{j=1}^{M} p_j\{1 - [1 - e^{-L_j(n_j/B_j)^{\alpha_j} T_j}]^{n_j}\},$$

subject to

$$\sum_{j=1}^{M} B_j = B.$$

Since the threshold values T_j's and the parameters L_j's are fixed, we can absorb L_j into T_j in the analysis without loss of generality. Therefore, we assume L_j to be equal to one. We also suppose that the research teams' achievement is very sensitive to resource allocation. In other words, we set $\alpha_j = 1,\ j = 1, 2, .., M$.

Therefore, the goal of maximizing the weighted sum of probabilities that the most successful team exceeds an activity-specific threshold value T_j translates into

$$\min_{\mathbf{B},\mathbf{n}} f(\mathbf{B},\mathbf{n},\mathbf{p}) \equiv \sum_{j=1}^{M} p_j[1-e^{-(n_j/B_j)T_j}]^{n_j}$$

subject to

$$\sum_{j=1}^{M} B_j = B$$

where $\mathbf{B} = (B_1, ..., B_M)$, $\mathbf{n}$ and $\mathbf{p}$ are defined similarly.

Before proceeding to the results, we present below an example with two activities. Let $B = 10$, $T_1 = T_2 = 8$. We plot the objective function for several choices of (n_1, n_2) as a function of B_1, when $L_1 = L_2 = 1$. We observe that the curve corresponding to $n_1 = n_2 = 1$ lies below all the others. This observation is formally proved below. Furthermore, the function for $n_1 = n_2 = 1$ is neither convex nor concave but has three local minima, two of which are both global minima, namely $B_1 = 0$, and $B_1 = 10$.

Insert Figure 1 about here

In the rest of the paper we use $\mathbf{1}$ to indicate a vector with all elements equal to one. For ease of notation, we define $r_j = T_j/B$ which can be interpreted as a scaled measure of the success threshold for activity j per unit budget. The higher this ratio gets, the smaller is the probability that an activity is considered successful. Hence, larger values of r_j indicate higher levels of ambition for that activity.

Theorem 1 *If* $r_j \geq \ln 2$ *for* $j = 1, \ldots, M$*,* $f(\mathbf{B}, \mathbf{n}, \mathbf{p})$ *is minimized at* $\mathbf{n} = \mathbf{1}$*, for all* $\mathbf{B}$*.*

Proof:
We first show that when $r_j \geq \ln 2$ for $j = 1, 2, .., M$, $f(\mathbf{B}, \mathbf{1}, \mathbf{p})$ is smaller than or equal to $f(\mathbf{B}, \mathbf{n}, \mathbf{p})$ for $\mathbf{n} \geq 1$ (that is, each component is greater than or equal to one), for all values of $\mathbf{B}$.

$$f(\mathbf{B}, \mathbf{1}, \mathbf{p}) = \sum_{i}^{M} p_i (1 - e^{-T_i/B_i})$$

Then, for any $\mathbf{n} \geq 1$, the difference $f(\mathbf{B}, \mathbf{n}, \mathbf{p}) - f(\mathbf{B}, \mathbf{1}, \mathbf{p})$ is given by

$$\sum_{i=1}^{M} p_i \left[(1 - e^{-(n_i T_i/B_i})^{n_i} - (1 - e^{-(T_i/B_i})\right].$$

Note that if $n_i = 1$ for some i, the differences corresponding to those i values will be zero and they will not contribute to the above sum.

Letting

$$\frac{T_i}{B_i} = x_i,$$

and omitting p_i which is a positive multiplicative constant, the i'th term in the previous expression is written as

$$(1 - e^{-n_2 x_i})^{n_i} - (1 - e^{-x_i}).$$

This expression is zero when $n_i = 1$, and otherwise its derivative with respect to n_2 is

$$(1 - e^{-n_i x_i})^{n_i} \left\{ \frac{n_i x e^{-n_i x_i}}{1 - e^{-n_i x_i}} + \ln(1 - e^{-n_i x_i}) \right\}.$$

This derivative vanishes when $n_i x_i = \ln 2$, it is negative if $n_i x_i < \ln 2$, and positive if $n_i x_i > \ln 2$. Then,

$$(1 - e^{-n_i x_i})^{n_i} - (1 - e^{-x_i})$$

is always nonnegative for all values of $n_i \geq 1$ if $x_i \geq \ln 2$. Thus, for all $\mathbf{B}$, $f(\mathbf{B}, \mathbf{1}, \mathbf{p}) \leq f(\mathbf{B}, \mathbf{n}, \mathbf{p})$ for $\mathbf{n} \geq 1$ if

$$\frac{T_i}{B_i} \geq \ln 2.$$

Since $0 < B_i < B$, a sufficient condition is given as

$$r_i \geq \ln 2.$$

Now suppose for some $i_1, i_2, ..., i_k$, $n_{i_j} = 0$ and the rest of the n_j's are 1. For simplicity let $i_1 = M-k+1, i_2 = M-k+2, ..., i_k = M$, so that the first $M-k$ elements of $\mathbf{n}$ are 1 and the rest are zero and denote the corresponding $\mathbf{n}$ vector by $\mathbf{n_k}$. Then

$$\begin{aligned} f(\mathbf{B}, \mathbf{n_k}, \mathbf{p}) \quad - & \quad f(\mathbf{B}, \mathbf{1}, \mathbf{p}) = \\ & \sum_{i=1}^{M} p_i \left((1 - e^{-(n_i T_i / B_i})^{n_i} - (1 - e^{-(T_i/B_i})\right) \\ = & \sum_{i=1}^{M-k} p_i (1 - e^{-T_i/B_i}) + \sum_{i=M-k+1}^{M} p_i - \sum_{i=1}^{M} p_i (1 - e^{-(T_i/B_i}) \\ = & \sum_{i=M-k+1}^{M} p_i e^{-(T_i/B_i)} \geq 0 \end{aligned}$$

Hence a sufficient condition for the theorem to hold is that

$$r_j \geq \ln 2 \equiv \min_j r_j \geq \ln 2$$

∎

We conclude from the above theorem that when $r_j \geq \ln 2$ for $j = 1, \ldots, M$, at most one research team in each activity can be funded. This result which may not seem intuitive at first sight can be attributed to the form of the objective function since the success of a team is measured by the tail probability of an exponential distribution. If the desired achievement rate per unit budget T_j/B is high the policy avoids parallel funding in order to maintain the success level expressed by this small probability. Hence, other objective functions can result in different policies which may encourage more parallel funding.

The following theorem complements the above result by providing the optimal budget allocation. Define $c_j = p_j/2^{a_j}$ for $j = 1, \ldots, M$ where $a_j = r_j/\ln 2$.

Theorem 2 *Let* $c^* = \max_j\{c_j\}$ *and* $\mathcal{J} = \{j | 1 \leq j \leq M$ *and* $c_j = c^*\}$. *Then for* $r_j \geq \ln 2$ *for* $j = 1, \ldots, M$*, the following holds:*

1. *If* $\mathcal{J}$ *is a singleton, then* $f(\mathbf{B}, \mathbf{1}, \mathbf{p})$ *is minimized at* $B_i = B$*, and* $B_j = 0$*, for all* $j \neq i$*, for which* $c_i = c^*$.
2. *If* $\mathcal{J} = \{i_1, i_2, \ldots, i_r\}$ *and if* $p_{i_1} = p_{i_2} = \ldots = p_{i_r} \equiv p$*, then* $f(\mathbf{B}, \mathbf{1}, \mathbf{p})$ *is minimized at* $B_{i_k} = B$*, and* $B_j = 0$*, for all* $j \neq i_k$*, for* $k = 1, 2, .., r$*. Furthermore, if it also holds that* $c^*/p = (1/r)^{1/r-1}$*, then* $f(\mathbf{B}, \mathbf{1}, \mathbf{p})$ *is also minimized at* $B_{i_k} = B/r$*, for* $k = 1, 2, .., r$ *and* $B_j = 0$*, for all* $j \neq i_k$.

Proof:
The original problem of minimizing $f(\mathbf{B}, \mathbf{1}, \mathbf{p})$ can be transformed into

$$\max_{B_1, \ldots, B_M} \sum_{j=1}^{M} p_j e^{-T_j/B_j} \tag{1}$$

subject to

$$\sum_{j=1}^{M} B_j = B.$$

Since $r_j \geq \ln 2$, we can write $r_j = a_j \ln 2$ for some $a_j \geq 1$ which implies $T_j = B a_j \ln 2$ and (1) can be written as

$$\max_{\mathbf{X}} f(\mathbf{p}, \mathbf{X}) \equiv \max_{X} \sum_{j=1}^{M} p_j \left(\frac{1}{2^{a_j}}\right)^{1/x_i} \tag{2}$$

where $\mathbf{X} = (x_1, x_2, ..., x_M)$ and $0 \leq x_i \leq 1$. First observe that

$$\lim_{x_i \to 1, x_j \to 0, j \neq i} f(\mathbf{p}, \mathbf{X}) = \frac{p_i}{2^{a_i}} = c_i$$

Then it is sufficient to show that the maximum in (2) $\leq \max_i\{c_i\} = c^*$. To see this we write

$$
\begin{aligned}
f(\mathbf{p}, \mathbf{X}) &= \sum_{j=1}^{M} p_j \left(\frac{1}{2^{a_j}}\right)^{1/x_i} \\
&= \sum_{j=1}^{M} \frac{p_j}{2^{a_j}} \left(\frac{1}{2^{a_j}}\right)^{(1-x_i)/x_i} \\
&\leq c^* \sum_{j=1}^{M} \left(\frac{1}{2^{a_j}}\right)^{(1-x_i)/x_i} \qquad (3) \\
&\leq c^* \sum_{j=1}^{M} \left(\frac{1}{2}\right)^{(1-x_i)/x_i} \qquad (4)
\end{aligned}
$$

Now using the method of Lagrange multipliers we can easily see that the R.H.S of (4) is maximized at $x_i = 1/M$ for $i = 1, 2, .., M$ and at this point its value is $c^* M(0.5)^{M-1} \leq c^*$ for all values of $M \geq 1$. This proves part 1.

For part 2, suppose $c^* = c_{i_1} = c_{i_2} = ... = c_{i_r}$ and $p_{i_1} = p_{i_2} = ... = p_{i_r} \equiv p$. These imply also that $a_{i_1} = a_{i_2} = ... = a_{i_r} \equiv a$. For ease of notation let $i_1 = 1, i_2 = 2, ..., i_2 = r$,for $r \leq M$. It is sufficient to check whether $f(\mathbf{p_1}, \mathbf{X_1}) = f(\mathbf{p_1}, \mathbf{X_2})$, with

$$
\begin{aligned}
\mathbf{p_1} &= (p, p, .., p, p_{r+1}, .., p_M) \\
\mathbf{X_1} &= (1/r, 1/r, .., 1/r, 0, .., 0) \\
\mathbf{X_2} &= (0, .., 1, ., 0, .., 0),
\end{aligned}
$$

where in X_2, the single entry 1 occurs anywhere in the first r entries. Then

$$
\begin{aligned}
f(\mathbf{p_1}, \mathbf{X_1}) &= \sum_{j=1}^{r} p \left(\frac{1}{2^{a_j}}\right)^{r} \\
&= p.r \left(\frac{1}{2^{a}}\right)^{r} \qquad (5) \\
f(\mathbf{p_1}, \mathbf{X_2}) &= p \left(\frac{1}{2^{a}}\right) \qquad (6)
\end{aligned}
$$

Then we see that (5) and (6) are equal if $1/2^a = c^*/p = (1/r)^{1/(r-1)}$. ∎

Having established in Theorem 1 that only a single team within each activity should be funded when $T_j/B \geq \ln 2, \forall j = 1, \ldots, M$, Theorem 2 states the following:

1. If there is a single activity that "dominates" in terms of its weight (c_j) then that activity receives all the available budget.
2. If several activities simultaneously "dominate" in terms of their weights, there are multiple optima where one of the dominating activities gets all the funding. Furthermore, if a certain condition on the maximum weight c^* holds, there is yet another optimum funding scheme where all activities equally share the available budget.

Note that part 2 of Theorem 2 always holds without the condition on c^*/p for $M = 2$. Also, it is enough to compute the c_i values and select the maximum to find the activity that receives all the funding. As a simple illustration of the above theorem, consider the example in Figure 1. Here $p_1 = p_2 = 1$, and we compute $c_1 = c_2 = 0.449329$. Hence, clause 2 of the theorem applies and we have two global minima, $(B_1 = 10, B_2 = 0)$ and $(B_1 = 0, B_2 = 10)$.

Maximizing the expected number of teams achieving a threshold. In the foregoing discussion we considered the probability that the most successful team within each activity exceeds a threshold and a weighted sum of these probabilities is aimed to be maximized. As an alternative objective, Gerchak [3] considers the problem of maximizing the expected number of teams attaining a pre-specified threshold (Problem 3 in the above reference). Since it is assumed that the teams within each activity works independently with an identical achievement distribution, the number of teams achieving a certain threshold will have a binomial distribution with parameters n_i and $p_i^* = [1 - F_j(T_j; B_j/n_j)]$, where $F_j(\cdot)$ corresponds to the achievement distribution of a team in activity $j, j = 1, \ldots, M$. If activities have different priorities reflected by p_j's as before, the objective function for $M = 2, \alpha_i = 1, i = 1, 2$ becomes:

$$\begin{aligned} & \max_{B_1, n_1, n_2} \left[p n_1 e^{-T_1 (n_1/B_1)} + (1-p) n_2 e^{-T_2 (n_2/(1-B_1))} \right] \\ \equiv \; & \max_{B_1, n_1, n_2} f(B_1, n_1, n_2) \end{aligned}$$

where B is taken to be one without loss of generality.

We briefly indicate below that the results obtained above also extend to this problem. In particular, we observe that $f(B_1, n_1, n_2) > f(B_1, n_1, 0)$ for all $n_1 \geq 1$ and $f(B_1, n_1, n_2) > f(B_1, 0, n_2)$ for all $n_2 \geq 1$. Further,

$$\begin{aligned} f(B_1, 1, 1) \quad - \quad & f(B_1, 1, n_2) \\ = \quad & (1-p)[e^{-T_2/(1-B_1)} - n_2 e^{-T_2 n_2/(1-B_1)}] \end{aligned}$$

For $n_2 \geq 2$, the above expression is always positive if $\ln 2 \leq T_2$, which follows from the arguments presented in the previous section. Similarly we can show that $f(B_1, 1, 1) - f(B_1, n_1, 1)$ is always nonnegative if $\ln 2 \leq T_1$ holds. Therefore the sufficient condition for $(n_1, n_2) = (1, 1)$ to be the optimal number of teams turns out to be that $\ln 2 \leq T_i, i = 1, 2$, similar to the previous problem. The problem of finding the optimal value of B_1 that maximizes $f(B_1, 1, 1)$ also reduces to the previous problem since

$$\begin{aligned} & \max_{B_1} \left\{ pe^{-T_1/B_1} + (1-p)e^{-T_1/(1-B_1)} \right\} \\ \equiv \quad & \min_{B_1} \left\{ p[1 - e^{-T_1(1/B_1)}] + (1-p)[1 - e^{-T_1(1/(1-B_1))}] \right\}. \end{aligned}$$

This seemingly counterintuitive result can again be explained by the tail probability expression involved in the objective function. On the other hand, the sufficient conditions of the above results may be stronger than necessary. That is, if this condition is not satisfied the optimal policy may still advocate funding a single team as exemplified in [3] Table 8.

3 Conclusion

In this paper, we obtained a sufficient condition under which a nonlinear resource allocation problem introduced by Gerchak and Kilgour [2] admits closed-form solutions. The cases where the sufficient condition fails to hold are deemed less interesting in applications by Gerchak and Kilgour [2]. Closed-form solutions in such cases are not possible as observed by Gerchak and Kilgour [2] through numerical experimentation since the optimal decision differs from one case to another.

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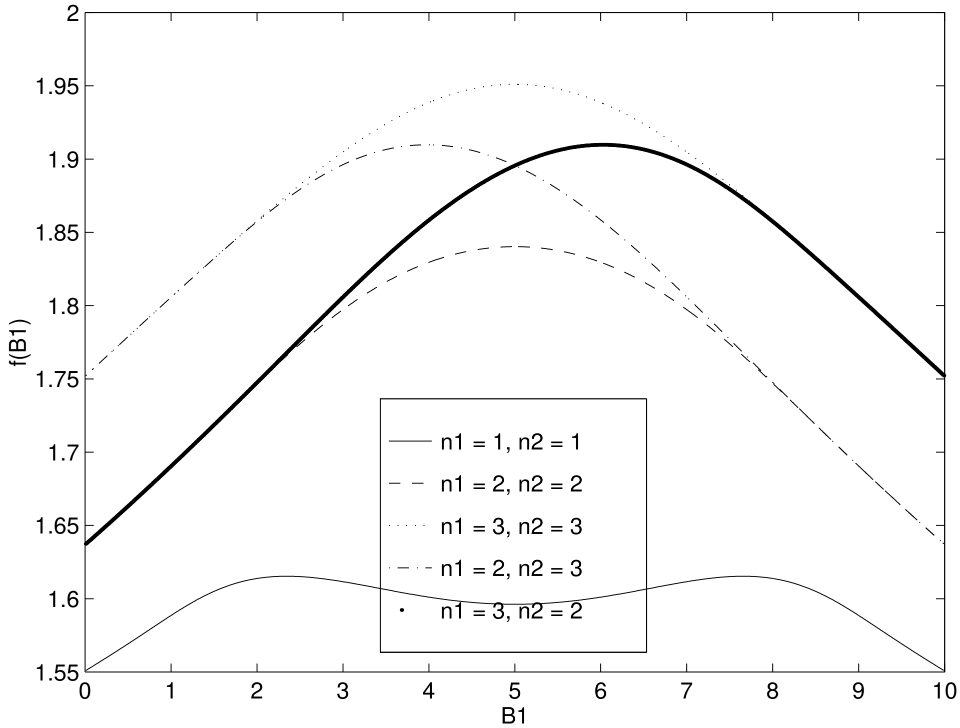

Figure 1: The plot of the function f on a two activity problem for different values of n_1, n_2.