15.053 Thursday, March 14

- Introduction to Network Flows
- Handouts: Lecture Notes

Network Models
- Linear Programming models that exhibit a very special structure
- Can use this structure to dramatically reduce computational complexity
- First widespread application of LP to problems of industrial logistics
- Addresses huge number of diverse applications

Notation and Terminology
Note: Network terminology is not (and never will be) standardized. The same concept may be denoted in many different ways.

Called:
- NETWORK
- directed graph
- graph

Class Handouts (Ahuja, Magnanti, Orlin)

Also Seen
- NETWORK
- digraph
- graph

Network $G = (N,A)$
- Vertex set $V = \{1,2,3,4\}$
- Arc set $A = \{(1,2),(1,3),(3,2),(3,4),(2,4)\}$

An Overview of Some Applications of Network Optimization

<table>
<thead>
<tr>
<th>Applications</th>
<th>Physical analog of nodes</th>
<th>Physical analog of arcs</th>
<th>Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Communication systems</td>
<td>phone exchanges, computers, transmission facilities, satellites</td>
<td>Cables, fiber optic links, microwave relay links</td>
<td>Voice messages, Data, Video transmissions</td>
</tr>
<tr>
<td>Hydraulic systems</td>
<td>Pumping stations, lakes, reservoirs</td>
<td>Pipelines</td>
<td>Water, Gas, Oil, Hydraulic fluids</td>
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<tr>
<td>Integrated computer circuits</td>
<td>Gates, registers, processors</td>
<td>Wires</td>
<td>Electrical current</td>
</tr>
<tr>
<td>Mechanical systems</td>
<td>Joints</td>
<td></td>
<td>Heat, Energy</td>
</tr>
<tr>
<td>Transportation systems</td>
<td>Intersections, Airports, Rail yards</td>
<td>Highways, Airline routes Railroads</td>
<td>Passengers, freight, vehicles, operators</td>
</tr>
</tbody>
</table>

Examples of terms.

Path: Example: 5, 2, 3, 4.
(or 5, c, 2, b, 3, e)
Note that directions are ignored.

Directed Path: Example: 1, 2, 3, 4
(or 1, a, 2, b, 3, e)
Directions are important.

Cycle or circuit (or loop)
1, 2, 3, 1. (or 1, a, 2, b, 3, e)
Note that directions are ignored.

Directed Cycle: (1, 2, 3, 4, 1) or 1, a, 2, b, 3, c, 4, d, 1
Directions are important.
More Definitions

A network is **connected** if every node can be reached from every other node by following a sequence of arcs in which direction is ignored.

A **spanning tree** is a connected subset of a network including all nodes, but containing no loops.

The Minimum Cost Flow Problem

Let $x_{ij}$ be the flow on arc $(i,j)$.

Minimize the cost of sending flow

\[ \text{s.t. } Flow\ out\ of\ i - \text{Flow}\ into\ i = b_i \]

\[ 0 \leq x_{ij} \leq u_{ij} \]

Minimize

\[ \sum_{i,j} c_{ij} x_{ij} \]

s.t.

\[ \sum_{j} x_{ij} - \sum_{i} x_{ij} = b_i \text{ for all } i \]

\[ 0 \leq x_{ij} \leq u_{ij} \text{ for all } i-j \]

Example Formulation

Min \[-3x_{12} + 8x_{13} + 7x_{23} + 3x_{24} + 2x_{34} \]

s.t.

\[ x_{12} + x_{13} = 4 \]

\[ x_{23} + x_{24} - x_{12} = 3 \]

\[ x_{34} - x_{13} - x_{23} = -5 \]

\[ -x_{24} - x_{34} = -2 \]

\[ 0 \leq x_{ij} \leq 6 \]

\[ 0 \leq x_{12} \leq 5 \]

\[ 0 \leq x_{23} \leq 2 \]

\[ 0 \leq x_{24} \leq 4 \]

\[ 0 \leq x_{34} \leq 7 \]

An Application of the Minimum Cost Flow Problem

Ship from suppliers to customers, possibly through warehouses, at minimum cost to meet demands.

Useful Facts About The Minimum Cost Flow Problem

- Suppose the following properties of the constraint matrix, $A$ (ignoring simple upper and lower variable bounds, such as $x \leq 7$) hold:
  1. all entries of $A$ are 0 or 1 or -1
  2. there is at most one 1 in any column and at most one -1.
- Then this is a minimum cost flow problem.
Useful Facts (cont’d)

Theorem. If one carries out the simplex algorithm on the minimum cost flow problem with integer valued capacities and RHS, then at every iteration of the simplex algorithm, each coefficient in the tableau (except for costs and RHS) is either 0 or -1 or 1.

Corollary. The optimal LP solution is integer valued.

The Minimum Cost Flow Problem

A network with costs, capacities, supplies, demands

Minimize the cost of sending flow
s.t. Flow out of i - Flow into i = bi
Flow on arc (i,j) ≤ uij

The Transportation Problem

Suppose that one wants to ship from warehouses to retailers

In this example:
3 warehouses
4 retailers
ai is the supply at warehouse i.
bj is the demand at retailer j.
cij is the cost of shipping from i to j.
There are no capacities on the arcs.
Let xij be the amount of flow shipped from warehouse i to retailer j.
How do we formulate an LP?

The Transportation Problem is a Min Cost Flow Problem

Minimize the cost of sending flow
s.t. Flow out of i - Flow into i = bi
0 ≤ xij ≤ uij
Flow out occurs at the supply nodes.
Flow in occurs at demand nodes.
Capacities are infinite: uij = ∞

Useful Facts About Transportation Problem

Suppose that
(1) the constraint matrix can be partitioned into
A1x = b1, and A2x = b2.
(2) all entries of A1 and A2 are 0 or 1
(3) there is at most one 1 in any column of A1 or A2
Then this is a transportation problem.

Theorem. If one carries out the simplex algorithm on the transportation problem, then at every iteration of the simplex algorithm, each coefficient in the tableau (except for costs and RHS) is either 0 or -1 or 1. The costs and RHS are both integer valued.

Corollary. The optimal solution to the LP is integer valued.
The Assignment Problem

Suppose that one wants to assign tasks to persons

<table>
<thead>
<tr>
<th>Tasks</th>
<th>Persons</th>
</tr>
</thead>
<tbody>
<tr>
<td>s(4) = 1</td>
<td>d(1) = 1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
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In this example:
- 4 tasks
- 3 persons
- No two tasks to the same person
- Each person gets a task
- \( c_{ij} \) is the "cost" of assigning task \( i \) to person \( j \).

Let \( x_{ij} = 1 \) if task \( i \) is assigned to \( j \).
Let \( x_{ij} = 0 \) otherwise.

How do we formulate an LP?

The Assignment Problem

In general the LP formulation is given as

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i} \sum_{j} c_{ij} x_{ij} \\
\text{subject to} & \quad \sum_{j} x_{ij} = 1, \forall i = 1, \ldots, n \\
& \quad \sum_{i} x_{ij} = 1, \forall j = 1, \ldots, n \\
& \quad x_{ij} = 0 \quad \text{or} \quad 1, \forall ij
\end{align*}
\]

Each supply is 1
Each demand is 1

More on the Assignment Problem

<table>
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The assignment problem is a special case of the transportation problem.

The simplex algorithm can solve the LP relaxation, and it will give integer answers, that is, it will solve the assignment problem.

An Application of the Assignment Problem

Suppose that there are moving targets in space. You can identify each target as a pixel on a radar screen. Given two successive pictures, identify how the targets have moved.

The Maximum Flow Problem

Network \( G = (N, A) \):
- Source \( s \) and sink \( t \)
- Capacities \( u_{ij} \) on arc \((i,j)\)
- Variable: Flow \( x_{ij} \) on arc \((i,j)\)

Graph with capacities

Maximize the flow leaving \( s \)

\[ \text{s.t.} \quad \text{Flow out of } i - \text{Flow into } i = 0 \text{ for } i \neq s, t \]

\[ 0 \leq x_{ij} \leq u_{ij} \]

This is not formulated as a special case of a minimum cost flow formulation.
Can we reformulate it in this way?
More on the maximum flow problem

Is the current flow optimal?
An s-t cut is a separation of the nodes into two parts S and T, with s in S and t in T.
The capacity of the cut is the sum of the capacities from S to T.
The max flow from s to t is at most the capacity of any s-t cut.

Graph with capacities and flows (underlined)

The Shortest Path Problem

What is the shortest path from an origin or source node (often denoted as s) to a destination or sink node, (often denoted as t)? What is the shortest path from node 1 to node 6?

Assumptions for now:
1. There is a path from node s to all other nodes.
2. All arc lengths are non-negative

Direct Applications

- What is the path with the shortest driving time from 77 Massachusetts Avenue to Boston City Hall?
- What is the path from Building 7 to Building E40 that minimizes the time spent outside?
- What is the communication path from i to j that is the fastest (taking into account congestion at nodes)?

The Shortest Path Problem

- Fact: The Shortest path problem is a special case of the minimum cost flow problem
- Lots of interesting applications (coming up)
- Very fast algorithm (coming up)
- Connection to dynamic programming (several lectures from now)

Formulation as a linear program

In general the LP formulation is given as

\[
\text{Minimize } \sum_{j=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\sum_{j=1}^{n} x_{ij} - \sum_{k=1}^{n} x_{kj} = \begin{cases} 
1, & i = s \\
-1, & i = t \\
0, & \forall i = 1, \ldots, n 
\end{cases} \\
x \geq 0, \forall ij
\]

Conclusions.

- Advantages of the transportation problem and the minimum cost flow problem
  - Integer solutions
  - Very fast solution methods
  - Extremely common in modeling
- Today we saw the following:
  - The minimum cost flow problem
  - The transportation problem
  - The assignment problem
  - The maximum flow problem
  - The shortest path problem