Putting Linear Programs into standard form
Introduction to Simplex Algorithm

Note: this presentation is designed with animation to be viewed as a slide show.

Linear Programs in Standard Form

1. Non-negativity constraints for all variables.
2. All remaining constraints are expressed as equality constraints.
3. The right hand side vector, b, is non-negative.

<table>
<thead>
<tr>
<th>LP not in Standard Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximize 3x₁ + 2x₂ - x₃ + x₄</td>
</tr>
<tr>
<td>x₁ + 2x₂ + x₃ - x₄ ≤ 5;</td>
</tr>
<tr>
<td>-2x₁ - 4x₂ + x₃ + x₄ ≤ -1;</td>
</tr>
<tr>
<td>x₁ ≥ 0, x₂ ≥ 0</td>
</tr>
</tbody>
</table>

Converting Inequalities into Equalities

Plus Non-negatives

To covert a “≤” constraint to an equality, add a slack variable.

Before                      After
x₁ + 2x₂ + x₃ - x₄ ≤ 5      x₁ + 2x₂ + x₃ + s = 5
s ≥ 0

s is called a slack variable, which measures the amount of “unused resource.”

Note that s = 5 - x₁ - 2x₂ - x₃ + x₄.

Converting “≥” constraints

- Consider the inequality -2x₁ - 4x₂ + x₃ + x₄ ≤ -1;
- Step 1. Eliminate the negative RHS
  2x₁ + 4x₂ - x₃ - s = 1
- Step 2. Convert to an equality
  2x₁ + 4x₂ - x₃ - s = 1
s ≥ 0
- The variable added will be called a surplus variable.

To covert a “≥” constraint to an equality, subtract a surplus variable.

More Transformations

How can one convert a maximization problem to a minimization problem?

Example: Maximize 3W + 2P
Subject to “constraints”
Has the same optimum solution(s) as

Minimize -3W -2P
Subject to “constraints”

The Last Transformations (for now)

Transforming variables that may take on negative values.

| Transforming x₁: replace x₁ by y₁ = -x₁;  y₁ ≥ 0. |
| max -3y₁ + 4x₂ + 5x₃ |
| -2y₁ - 5x₂ + 2x₃ = 17 |
| y₁ ≥ 0, x₂ is unconstrained in sign, x₃ ≥ 0 |

One can recover x₁ from y₁.
Transforming variables that may take on negative values.

\[
\begin{align*}
\text{max } & -3y_1 + 4x_2 + 5x_3 \\
& -2y_1 - 5x_2 + 2x_3 = 17
\end{align*}
\]

all vars \(\geq 0\)

Transforming \(x_3\): replace \(x_3\) by \(x_3 = y_3 - y_2\); \(y_2 \geq 0, y_3 \geq 0\).

One can recover \(x_2\) from \(y_2, y_3\).

Another Example

- Exercise: transform the following to standard form (maximization):

\[
\begin{align*}
\text{Minimize } & \quad x_1 + 3x_2 \\
\text{Subject to } & \quad 2x_1 + 5x_2 \leq 12 \\
& \quad x_1 + x_2 \geq 1 \\
& \quad x_1 \geq 0
\end{align*}
\]

Perform the transformation with your partner

Preview of the Simplex Algorithm

\[
\begin{align*}
\text{maximize } & \quad -3x_1 + 2x_2 \\
\text{subject to } & \quad -3x_1 + 3x_2 + x_3 = 6 \\
& \quad -4x_1 + 2x_2 + x_4 = 2 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

Optimality Conditions Preview

- Exercise: transform the following to standard form (maximization):

\[
\begin{align*}
\text{Minimize } & \quad x_1 + 3x_2 \\
\text{Subject to } & \quad 2x_1 + 5x_2 \leq 12 \\
& \quad x_1 + x_2 \geq 1 \\
& \quad x_1 \geq 0
\end{align*}
\]

Perform the transformation with your partner

Preview of the Simplex Algorithm

\[
\begin{align*}
\text{maximize } & \quad z = -3x_1 + 2x_2 \\
\text{subject to } & \quad -3x_1 + 3x_2 + x_3 = 6 \\
& \quad -4x_1 + 2x_2 + x_4 = 2 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

Optimality Conditions Preview

\[
\begin{align*}
\text{maximize } & \quad z = -3x_1 + 2x_2 \\
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& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

Obvious Fact: If one can improve the current basic feasible solution \(x\), then \(x\) is not optimal.

Idea: assign a small value to just one of the non-basic variables, and then adjust the basic variables.
The current basic feasible solution (bfs) is not optimal!

<table>
<thead>
<tr>
<th>-2</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

If there is a positive coefficient in the z row, the basis is not optimal.

Recall: \( z = -3x_1 + 2x_2 \)

Let \( x_2 = \Delta \). How large can \( \Delta \) be? What is the solution after changing \( x_2 \)?

\[
\begin{array}{c|cccc}
  -2 & x_1 & x_2 & x_3 & x_4 \\
  1   & -3 & 2   & 0   & 0   \\
  0   & -3 & 3   & 1   & 0   \\
  0   & -4 & 2   & 0   & 1
\end{array}
\]

\[
\begin{array}{c|cccc}
  x_3 = 6 - 3\Delta \\
  x_4 = 2 - 2\Delta \\
  z = 2\Delta
\end{array}
\]

What happens to \( x_1, x_4 \) and \( z \)?

Optimality Conditions

(note that the data is different here)

<table>
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<tr>
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<tbody>
<tr>
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<td>-4</td>
<td>0</td>
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<td>1</td>
<td>0</td>
</tr>
<tr>
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<td>2</td>
<td>0</td>
<td>1</td>
</tr>
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</table>

Important
Fact. If there is no positive coefficient in the z row, the basic feasible solution is optimal!

\[
\begin{array}{c|cccc}
  z = -2x_1 + 4x_2 + 8.
\end{array}
\]

Therefore \( z \leq 8 \) for all feasible solutions. But \( z = 8 \) in the current basic feasible solution. This basic feasible solution is optimal!

Summary of Simplex Algorithm

- Start in canonical form with a basic feasible solution
- Check for optimality conditions
- If not optimal, determine a non-basic variable that should be made positive
- Increase that non-basic variable, and perform a pivot, obtaining a new bfs
- Continue until optimal (or unbounded).

OK. Let’s iterate again. \( z = x_1 - x_2 + 2 \)

\[
\begin{array}{c|cccc}
  -2 & x_1 & x_2 & x_3 & x_4 \\
  1   & 1   & 0   & 0   & -1 \\
  0   & 3   & 0   & 1   & -1.5 \\
  0   & -2  & 1   & 0   & .5
\end{array}
\]

The cost coefficient of \( x_1 \) is positive. Set \( x_1 = \Delta \) and \( x_4 = 0 \).

How large can \( \Delta \) be?

\[
\begin{array}{c|cccc}
  x_1 = \Delta \\
  x_2 = 1 + 2\Delta \\
  x_3 = 3 - 5\Delta \\
  x_4 = 0 \\
  z = 2 + \Delta
\end{array}
\]
A Digression: What if we had a problem in which $\Delta$ could increase to infinity?

$$z = x_1 - x_2 + 2$$

Set $x_1 = \Delta$ and $x_4 = 0$.

How large can $\Delta$ be?

End Digression: Perform another pivot

What is the largest value of $\Delta$?

Summary of Simplex Algorithm Again

- Start in canonical form with a basic feasible solution
  1. Check for optimality conditions
  - Is there a positive coefficient in the cost row?
  2. If not optimal, determine a non-basic variable that should be made positive
  - Choose a variable with a positive coefficient in the cost row.
  3. Increase that non-basic variable, and perform a pivot, obtaining a new bfs
  - We will review this step, and show a shortcut
  4. Continue until optimal (or unbounded).

Performing a “Pivot”. Towards a shortcut.

$$z = 2x_1 + 3$$

Exercise: to do with your partner.

More on performing a pivot

- To determine the column to pivot on, select a variable with a positive cost coefficient
- To determine a row to pivot on, select a coefficient according to a minimum ratio rule
- Carry out a pivot as one does in solving a system of equations.
Next Lecture

- Review of the simplex algorithm
- Formalizing the simplex algorithm
- How to find an initial basic feasible solution, if one exists
- A proof that the simplex algorithm is finite (assuming non-degeneracy)