IE 513: Linear Programming

Term Projects

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Images From a Portfolio

by N. Beḡen and G. Kurtay

1 Verbal Description of the Model

The typical forward problem is to find a solution to an equation that is satisfied over a domain, given that we know the boundary conditions, whereas the inverse problem is to find the boundary conditions that would have lead to a known solution. Inverse problems have useful application for risk management and pricing. This approach can be applied to the portfolio optimization problem to examine it in two dimensions. The standard portfolio optimization problem is to find the best possible portfolio when the possible states of the world and distributions for different assets are given. In the inverse portfolio optimization problem, it is intended to find the *implied views* (states of the world) that must occur for a given portfolio to be optimal.

Two problems will be considered here. The first problem is, using inverse portfolio optimization approach to obtain implied market parameters from traded instruments (assets). The inverse problem turns the roles of inputs and outputs of the forward optimization model around so, to consider the forward problem first will make everything clear. The forward portfolio optimization problem is to find the set of positions in a given, finite set of instruments that would maximize or minimize some given objective function when we are given a distribution of some market prices and liquidity constraints and market factors (yields,prices,etc.) in the future. The optimality criteria may be risk minimization, profit maximization, or more generally, an utility function that combines risk and profit in a meaningful way. Rational investors want to construct portfolio that simultaneously maximize their expected profit while minimizing their risk, given by the variance of the portfolio. So the portfolio selection problem has a two-dimensional (multi-criteria) nature that can be expressed by using the mean-variance portfolio optimization approach by Markowitz. This approach constructs an $efficient\ frontier$, a continuum of optimal portfolios that maximize their expected returns while simultaneously minimize their risk, given by their standard deviation, over a given investment horizon. The output of the model is a set of optimal portfolios, expressed as positions in each asset. The approach assumes that the distribution of all the asset returns is joint normal and given. Once the frontier is constructed, a single portfolio may be found that maximizes some utility function representing the investor's preferences.

The inverse portfolio optimization problem is to find the market conditions, namely the implied views, of a known chosen portfolio that make it optimal; we want to find those conditions under which no other feasible portfolio would perform better. If we look at it from Markowitz's point of view, it is finding the distribution parameters (means and covariance matrix) . These views can only be implied if we assume that investors behave as rational ones as expressed above and want to maximize utility. We focus on the problem of implying various parameters from the holdings of a single portfolio.

These views allow portfolio and risk managers to determine whether a portfolio or a strategy is consistent with some existing view on future movements of all the relevant factors. So the implied views may reflect possible asset returns distributions (in terms of their expectations,volatilities and correlations), investor preferences as parameters of a utility function, and liquidity assumptions. In general, because of the complexity of the problem , inverse and forward portfolio optimization problems do not present a unique solution, so the implied views are usually given on expected returns only, assuming everything else constant. Since the problem does not present a unique solution, the goal should be the recovery of useful information about the solution while recognizing the fundamental non-uniqueness of the problem. We can achieve this if we impose some priori restrictions on the solution set. The solution arises when we fix the covariance matrix and the Lagrange multipliers.

The second problem that will be considered is *Scenario Optimization*. Like the portfolio optimization problem, scenario optimization problem can be solved as a forward problem and inverse problem. In scenario optimization, a single period where only a finite number, s, of events(scenarios) could occur is considered. At the start of the period there is uncertainty as to which scenario will occur but at the end of the period exactly one of these events occur. Each scenario may be a single realization of all the relevant risk factors in the market. It is assumed that investors in the market have perception of the likelihood, p, of these scenarios. To make the problem more realistic, it is assumed that the set of possible future market events is finite and given. At the beginning, a portfolio called *Benchmark portfolio* is formed with only a finite number, n, of assets. Each of these assets have finite liquidity. The price of the benchmark at the beginning of the period is known. Depending upon which scenario actually occurs, the value of the benchmark at the end of the period is evaluated and price vector is found for s scenarios. Each element of this vector is named as *target value* for each scenario. This process is called *replication*. Each scenario must be transformed to a realization of the values of all assets and the benchmark. When the relationship between the factors and the assets is known, this can be accomplished through a set of pricing functions. The values of the assets can be compared with the target values. So the forward problem requires the set of scenarios, scenario probabilities, the prices of the assets and target values today, parameter λ which is a parameter related to the investor's risk preferences and liquidity assumptions.

The objective of forward scenario optimization is to find a replicating portfolio that behaves identically to the benchmark under all scenarios. This kind of a portfolio is called a perfect replication. We want to minimize deviations from this perfect replication. Deviations from the benchmark is called *benchmark risk*. The expected difference between the portfolio and the benchmark, *regret*, can be used to measure this risk.

When the inputs and outputs of the problem are reversed, the inverse problem is considered. The problem turns into finding the *scenario implied views* assuming liquidity and everything else constant. In other words, for a given portfolio, we try to find the set of scenario probabilities, p, which make the portfolio optimal. We can also obtain implied views for liquidity, risk aversion, current prices, or any mix.

It is important to remember the non-unique behavior of inverse problems so the set of meaningful solutions that provide useful information should be considered. Some criteria can be used to help us visualize the results. If we consider the set of probabilities P, that satisfy the inverse equations; $p \in P$ if and only if it is a possible implied view . Such a criteria is called *closest to prior* and in this case we want to find the implied views that are closer to a prior, specified solution. Suppose that we have a prior scenario probability distribution p^* for the scenario set. We can formulate it like;

$$min_p \|p - p^*\|$$

$$\text{s.t} \quad p \in P$$

Sensible choices for the norm are the $l_1$norm and the infinity norm which are LPs and $l_2$norm which is a QP.

Scenario optimization is a powerful and robust methodology for portfolio replication that can be used for asset and capital allocation, benchmark tracking, pricing and hedging in both complete and incomplete markets, and portfolio compression.The technique naturally accommodates non-normal distributions and non-linear instruments in addition to explicitly modeling discrete markets where trading may be costly and liquidity limited.

2 Mathematical Description of the Model

The Forward Scenario Optimization Model

Parameters
c= The price of the benchmark at the beginning of the period
$\mathbf{q} = (q_1, q_2, ...q_n)^T$ the known price of the assets at the start of the period
D is an s by n matrix that contains the values of every asset for each scenario at the end of the period.
$\mathbf{d_{ij}}$ = the value of the asset j for the i^{th} scenario
$\tau = (\tau_1, \tau_2, \tau_3, ...\tau_s)^T$, the price of the benchmark at the end of the period for each scenario
λ= risk aversion parameter

Decision Variables
$\mathbf{x_i}$ = the amount of the assets in the portfolio.
$\mathbf{y^+}$ = the positive deviations from the value of benchmark at the end of the period
$\mathbf{y^-}$ = the negative deviations from the value of benchmark at the end of the period

We choose *regret* as a measure of expected difference between the portfolio and the benchmark. If we denote regret by R, it is given by;

$$R = ||Dx - \tau||$$

if we use l_1-norm , then regret is the expectation of these differences under the measure p;

$$R = p^T|Dx - \tau|$$

we can denote the vector of negative and positive deviations from the benchmark by;
$y^- = (Dx - \tau)^- = min(0, Dx - \tau)$ and,
$y^+ = (Dx - \tau)^+ = max(0, Dx - \tau)$ then $R = p^T(y^+ + y^-)$ and

it is better to consider the underperformance from the benchmark;

$$R^- = p^T y^-$$

The rational investor wants to construct portfolio that simultaneously maximize their expected profit while minimizing their risk, so the optimal portfolios can be found by solving the program;

$$MR(k) = min_x R^-(x) \quad \text{s.t.}\, E_p\{\Pi(x)\} \geq k;\ \text{trading constraints}$$

This program is parameterized by k which is a lower bound on expected excess profit. If the value of the portfolio equals that of the benchmark today, the expected profit is simply given by

$$E_p\{\Pi(x)\} = p^T(y^+ - y^-)$$

If today the value of the portfolio and the benchmark are not equal, the expected excess profits can be expressed as

$$E_p\{\Pi(x)\} = (c - q^T x) + W_T p^T (y^+ - y^-))$$

We can formulate the same problem with a different way and use the utility function which is linear and get the resulting program;

$$U^*(x;\lambda) = max_x \quad E_p\{\Pi(x)\} - \lambda R^-(x), \quad \lambda \geq 0$$
$$\text{s.t., trading constraints}$$

Here instead of k, parameterization is in terms of λ which is a parameter related to the investor's risk preferences. We can insert the expressions that are shown above for regret and profit and utility becomes;

$$U(\lambda) = c - q^T x + W_T p^T y^+ - \gamma p^T y^- \quad \gamma \geq W_T \quad \gamma = \lambda + W^T$$

Finally, we get the program which gives us the efficient frontier and a single portfolio may be found that maximizes some utility function representing the investors preferences.

$$\begin{array}{llll}
U^*(\gamma) - c = \max_{x,y^+,y^-} & -q^T x + p^T y^+ - \gamma p^T y^- & & (\mathbf{EQ1}) \\
\text{s.t.} & Dx - y^+ + y^- & = \tau & (\mathbf{EQ2}) \\
 & A_L x & \leq b_L & (\mathbf{EQ3}) \\
 & x\ \text{free}, y^+, y^- & \geq 0 & (\mathbf{EQ4})
\end{array}$$

for some value $\gamma \geq 1$

In (EQ2), Dx is the value of the assets at the end of the period for each scenario, so with positive or negative deviations it must be equal to the price of the benchmark at the end of the period. (EQ3) is the liquidity constraint, which is a linear constraint added for completeness. We want to find the solution when γ is fixed.

The solution of the problem x^*,y^{*+},y^{*-} is uniquely defined by (EQ2). The verification of the feasibility of the portfolio amounts to the constraint (EQ3).

The Inverse Scenario Optimization Model

Decision Variables

p= We are interested in the implied probabilities so now p is an unknown variable.

π = The dual variable corresponding to (EQ2) of the primal problem

υ = The dual variable corresponding to (EQ3) of the primal problem

From the basic duality results we can attain simple solutions for solving scenario inverse problems. When we take the dual of the above linear program we get the following;

$$\begin{array}{lll}
(\max_{\pi} & \tau^T \pi - b_L^T \upsilon) & \\
\text{s.t.} & D^T \pi - A_L^T \upsilon & = q \\
 & \pi & \geq p \\
 & \pi & \leq \gamma p \\
 & \pi \text{ free}, \upsilon & \geq 0
\end{array}$$

We will benefit from complementary slackness theorem here. If we let x^* be a feasible solution of the primal problem, then x^* can only be optimal if there is some feasible solution μ^* for the dual problem. So the pair (x^*, μ^*) should satisfy complimentary slackness conditions. So we look at the constraints where slack variables are added.

$$
\begin{aligned}
(A_L x - b_L)v &= 0 \\
y^{+}(\pi - p) &= 0 \\
y^{-}(\pi - \gamma p) &= 0
\end{aligned}
$$

The objective here is to find the set of scenario probabilities, p, for which a given portfolio $x^* = (x_1^*, ..., x_n^*)$ is optimal. The following equations are attained.

$$
\begin{aligned}
D^T\pi - A_L^T v &= q \quad &(\mathbf{EQ5}) \\
\pi - p &\geq 0 \quad &(\mathbf{EQ6}) \\
\pi - \gamma p &\leq 0 \quad &(\mathbf{EQ7}) \\
e^T p &= 1 \quad &(\mathbf{EQ8}) \\
(A_L x^* - b_L)v &= 0v \quad &(\mathbf{EQ9}) \\
y^{*+}(\pi - p) &= 0 \quad &(\mathbf{EQ10}) \\
y^{*-}(\pi - \gamma p) &= 0 \quad &(\mathbf{EQ11}) \\
v &\geq 0 \quad &(\mathbf{EQ12}) \\
p &\geq 0 \quad &(\mathbf{EQ13})
\end{aligned}
$$

(EQ5), (EQ6), (EQ7) are the dual constraints of the primal problem, each of which corresponds to the coefficients of the primal objective function. (EQ8)is the constraint introduced because the probabilities sum up to 1. (EQ9), (EQ10), (EQ11), are the optimality conditions that satisfy complementary slackness. (EQ12), (EQ13) are the nonnegativity constraints.
We may get rid of constraint of type 0=0 so the final form of the linear

program is;

$$D^T\pi - A_L^T\upsilon = q \qquad \textbf{(EQ14)}$$

$$\pi_i - p_i = 0, for\{i|(y_i^{*+} > 0)\} \qquad \textbf{(EQ15)}$$

$$\pi_i - \gamma p_i = 0, for\{i|(y_i^{*-} > 0)\} \qquad \textbf{(EQ16)}$$

$$\pi_i - p_i \geq 0, \pi_i - \gamma p_i \leq 0, for\{i|(y_i^{*+} > 0), (y_i^{*-} > 0\} \qquad \textbf{(EQ17)}$$

$$e^T p = 1 \qquad \textbf{(EQ18)}$$

$$\upsilon^k = 0, for\{k|(A_L x^* - b_L < 0\} \qquad \textbf{(EQ19)}$$

$$\upsilon \geq 0 \qquad , \qquad \textbf{(EQ20)}$$

$$p \geq 0 \qquad \textbf{(EQ21)}$$

With unknowns π, υ, p

Objective Function
Because of the non-unique behavior of the inverse problems, some criteria should be used to get meaningful results so we can use here ‘closest to prior‘ criteria. The implied probabilities that are closest to a prior scenario distribution, p^* can be found by solving a linear programming problem. We can define deviations from p^* ;

$$p^+ = max(p - p^*, 0)$$
$$p^+ = max(p - p^*, 0)$$

We can rewrite criterion like;

$$min \sum_i p_i^+ + \sum_i p_i^- \qquad \textbf{(EQ22)}$$

because we want to minimize deviations from p^*.

Finally, (EQ14)-(EQ21) and objective function (EQ22) give us the closest implied scenario probabilities to the prior.

3 Discussion

The key to general scenario optimization approach is to create a set of scenarios: if the problem for each scenario is a linear program, then the linear programs can be combined into one large problem. The smaller linear programs are linked by variables corresponding to decisions that occur before it is known which scenario is active. One drawback of this approach is that the combined linear program can get very large if there are a lot of scenarios. For instance, suppose we think that demand for low and high machines is best modeled by a pair of random variables with a given correlation (say demand had correlation .5) that are uniformly distributed between 1000 and 3000. Then there are an infinite number of scenarios, which leads to a very large linear program! Scenario based optimization can still be used in this situation, however. The key is to generate a finite number of possible outcomes and optimize the decisions over those possibilities. In this case we could generate 100 different low and high demand pairs, and find the best decision relative to those scenarios. We can use heuristic decision making in this case.

Scenario optimization approach is getting much more attention as financial portfolios are being created in this way, because scenario optimization is a powerful tool for financial modeling. Applications of this robust technique include asset and capital allocation, pricing and hedging in complete and incomplete markets, risk-reward analysis, and estimation of implied interest rate curves and option volatilities. It also provides a useful,operational framework for explaining basic financial theory. General non-normal,discrete and subjective distributions and non-linear instruments, such as options can also be modeled with this approach.

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Habitat Dispersion in Forest Planning

by S. Gören and B. Pandül

1 Verbal Description of the Model

Timber is a scarce resource, which is only obtained from forests. But forests are not only for timber harvesting. Many animals live in forests. Cutting trees carelessly may harm the ecological balance, which results in undesirable situations. Cutting trees appropriately is a major topic in forest sciences, *forest planning.*

A plan usually describes the goals of the owner, and a schedule of activities for the woods or portion of the woods. Generally the larger the woodland acreage and the more different conditions of soils and tree sizes in the woods, the more details are needed. Where there are major differences in various sections of the woods, the needed forestry activity will likely be quite different. Therefore, the plan or activity will vary for each different major condition found; dividing the woods into sections or compartments allows the owner to treat them separately, for example, divisions or compartments in the woods may be the separation of large or mature timber areas from areas having only pole size timber or an old field just restocking with pine or hardwood sapling or poles.

A good plan should also take habitat dispersion objectives into account. The dispersion concern reflects spatial distribution or scattering of age classes needed to maintain adequate habitats for wildlife. Big animals that use recently regenerated areas to obtain forage will be reluctant to feed in areas farther than 600 feet from cover. Scenic beauty may also be a reason for the dispersion constraints

The operations research models that try to suggest a good plan are generally reduced to a *cutting stock problem.* This problem has been studied by many people in the literature. As in many discrete optimization problem the main difficulty is to handle medium or large-sized problems. The approach presented in this paper is similar to that of Gilmore and Gomory (1961). In theory they have to solve an integer program with exponentially many variables. Instead they solve a master linear program where every column is obtained by solving a subproblem. When they obtain an optimal solution of the master problem They use it to derive a heuristic integral solution. The subproblem is also an integer problem; it is the *maximum stable set problem,* a classic problem in combinatorial optimization.

Our objective is to define a cutting pattern of a forest that would respect the habitat of wildlife. This leads to specifications of the form and size of cutting units. To model the problem, they define a zone, which could be the whole forest, for which global requirements, such as minimum timber production or area with mature timber (cover), are defined. Within each zone there exists homogeneous management areas for which management alternatives are proposed. Each management area is divided into units, defined in terms of the dispersion problem. A management alternative for each area consist of a set of decisions, one for each unit of the area. These decisions specify whether cutting is performed in the unit. When they do these decisions there are some restrictions naturally: they should satisfy timber production or area with cover requirements.

2 Mathematical Description of the Model

2.1 Decision Variables:

It is required that no timber cutting is carried out in neighboring units in the same period. Thus they define the variable x as

$$x_{ij} = \begin{cases} 1 & \text{If area } i \text{ is managed with alternative } j, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Parameters:

a_{ij}^t: Total timber production of area i in period t , if alternative j is chosen;
l_{ij}^t: Number of hectares of cover in area i , in period t , if alternative j is chosen;
c_{ij}: Net present value of benefit associated to managing area i with alternative j;
f^t: Minimum required timber production in period t;
d_r^t: The minimum number of hectares of cover required for a zone r in period t;

The formulation is the following:

$Maximize \sum c_{ij} x_{ij}$

subject to

$$\sum_j x_{ij} = 1 \qquad \forall i$$
$$\sum_i \sum_j a^t_{ij} x^t_{ij} \geq f^t \qquad \forall t$$
$$\sum_{i \in Z_r} \sum_j l^t_{ij} x_{ij} \geq d^t_r \qquad \forall r, t$$
$$x_{ij} \in \{0, 1\}$$

The first set of constraints implies that one and only one alternative will be chosen for each area. The second set of constraints ensures that the production in each area satisfies the minimum requirement for that area. In the third set of constraints they denote by Z_r the set of areas in zone r. This set represents a minimum requirement of mature standing timber for each zone and period. Since for every area the number of managing alternatives is an exponential function of the number of units, this is an integer program with exponentially many variables. They shall replace the last set of constraints by

$$0 \leq x_{ij} \leq 1,$$

and solve this linear program with a column generation approach.

2.3 The solution method:

Since the above linear program is large in size, they use column generation method. They assume that they have a feasible basis of above master problem. Let $\alpha_i, \beta_t, \gamma^t_r$ be the dual variables associated with the constraint sets. If $\beta_t > 0, \gamma^t_r > 0$, than a slack variable associated with the second or third constraints enters the basis. In what follows they assume that $\beta \leq 0$ and $\gamma \leq 0$. A variable x_{ij} enters the basis if

$$\bar{c}_{ij} = c_{ij} - \alpha_i - \sum_t a^t_{ij} \beta_t - \sum_{\{r,t: i \in Z_r\}} l^t_{ij} \gamma^t_r > 0$$

To find (i, j) satisfying above reduced cost requirement can be modeled as a stable set problem in a graph. This resulting problem is NP-hard. Roughly speaking, it means that it is very unlikely that one could design a polynomial algorithm to solve it. To solve this problems they used two approaches. First they use a greedy heuristic and if it is not successful they use linear programming techniques.

If their heuristic fails to find a candidate to enter the basis they use the following approach:

Given a graph $G = (V, E)$, and a stable set $S \subseteq V$, the index vector of $S, x^S \in \mathbf{R}^{\text{v}}$ is defined by

$$x^S(u) = \begin{cases} 1 & \text{If } u \in S \\ 0 & \text{otherwise.} \end{cases}$$

The *stable set polytope of* G, denoted by $P(G)$, is the convex hull of the incidence vectors of the stable sets of G. Thus given a weight function $w : V \to \mathbf{R}$, the stable problem can be formulated as a linear program.

Maximize wx
subject to
$x \in P(G)$.

We need the system of inequalities that defines the above polytope which is full dimensional. This implies that (up to multiplication by a positive constant) there is a unique nonredundant inequalities system $Ax \leq b$ such that $P(G) = \{x : Ax \leq b\}$. These inequalities define the facets of the polytope. Although it is very difficult to find complete characterization of the facets, in many instances a partial description has been sufficient to solve the problem.

They denote by uv the edge whose endnodes are u and v. If $U \subseteq V$, then $E(U)$ denotes the set of edges with both endnodes in U, and $(U, E(U))$ is the *subgraph induced by* U. An odd cycle with no chord is called an *odd hole*. A maximal complete subgraph is called a *clique*. If G is a graph and u a node, we denote by $G\backslash u$ the graph obtained by deleting u. They denote by K_n the complete graph obtained with n nodes.

To formulate the stable set problem, the two most natural sets of inequalities are

$$\begin{array}{ll} x(u) \geq 0 & \forall u, \\ x(u) + x(v) \leq 1 & \forall uv. \end{array}$$

The first set of inequalities define facets of $P(G)$. An inequality of the second set defines a facet only if the edge uv is not included in a clique, otherwise it is dominated by a clique inequality.
If K is a clique, then the inequality

$$\sum_{u \in K} x(u) \leq 1,$$

defines a facet of $P(G)$. This is called a *clique inequality.*

If H is an odd hole then the inequality

$$\sum_{u \in H} x(u) \leq \frac{|H| - 1}{2},$$

is called an *odd hole inequality* and defines a facet of $P(H)$. This inequality is valid for $P(G)$, if it does not define a facet, then a strengthening of it defines a facet. The strengthening procedure is beyond our scope, in terms of linear programming. The procedure they use to solve the linear program to define the stable set is as follows: They solve a linear program with the nonnegativity constraints and the clique inequalities. Because of the special structure of the graphs in this study, it is easy to enumerate all their cliques. If the solution is integer, then they are done. (This was the case in 98% of the problems, as they report). Otherwise, they look for an odd hole inequality that is violated by this fractional vector; they lift it, they add it to the linear program and solve it again. Now, the problem they had at hand is a sequence of shortest path problems. If this approach results in failure, then they use classical branch and bound type algorithm to solve integer programs. They, however, report that they did not have to branch at all in this study. Therefore, linear programming was sufficient to solve the problem in its entire depth. By now, the master linear program has been solved, and they use a heuristic to obtain the integer solution to it, which is not of interest for our purposes.

3 Discussion

We see that linear programming can be very useful not just in the sense that it can be applied to an area different than engineering or management science, which we saw numerous examples, but also it proves to be a useful technique in the solution of problems such as stable set problem, which is known to be NP-hard, especially when one derives the advantage of special structure of the problem at hand.

In this study, the authors use linear programming for two different purposes. First, they solve a master linear program instead of an integer program with exponentially many variables. And second, they use column generation techniques to solve this linear program due to its large size. This column generation procedure turns out to be a member of one of the most difficult problem classes we know, NP-hard. Linear programming comes to rescue again. It may seem that the price we pay to escape computational burden of NP-hardness should be high, in terms of accuracy but it is not the case. The authors report that they have carried out thousands of experiments with their methods and have got satisfying results both in terms of computational time, and in terms of accuracy of the results.

Finally, it is surprising to see that we can take qualitative constraints into account along with the quantitative ones by linear programming . The authors model a forest planning that respects the habitat of wildlife and beauty of scene. These are clearly qualitative variables and usually need the judgment of decision makers. This model incorporates this type of constraints quite neatly. It is more convincing to get results from a mathematical model rather than to trust a decision maker's subjective judgment for us. To have the maximum net present benefit knowing that you do not harm the nature is very satisfying.

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The Role of Linear Semi-Infinite Programming in Signal-Adapted QMF Bank Design

by B. Gülbahar and M. Akbaş

1 Introduction

The structure of a two channel filter bank which is shown in Figure 1 was first introduced in mid-seventies by Croisier et al. [1], and has been studied by many researchers since that time. Therefore today there is a very strong underlying theory. This structure is used in many applications such as frequency division multiplexing, time division multiplexing, denoising, adaptive system design, feature detection, evaluation of Fast Discrete Wavelet Transform, etc. However the most famous and well-known application is sub-band coding such as audio, image, video and ECG compression. This work focuses on signal-adapted sub-band coding. One important point about that structure is that it enables multirate signal processing and operates like human auditory and visual system which are the optimal speech and image processors respectively. Researches showed that in human brain different frequency bands are processed by different parts of the brain. This is simply what a two-channel filter bank does. To understand how much powerful sub-band coding is FBI is a good example. Today FBI uses digitized finger prints which are compressed using *wavelet scalar quantization algorithm*(WSQ) [2]. At first it was expected that *discrete cosine transform* (DCT) which is the standard for famous JPEG images would win but it did not happen. At higher compression ratios such as 15:1 the blocking by DCT was severe. Therefore WSQ algorithm won the competition. In the following sections first a review of signal and systems is going to be given then mathematical modeling and LP formulation will come.

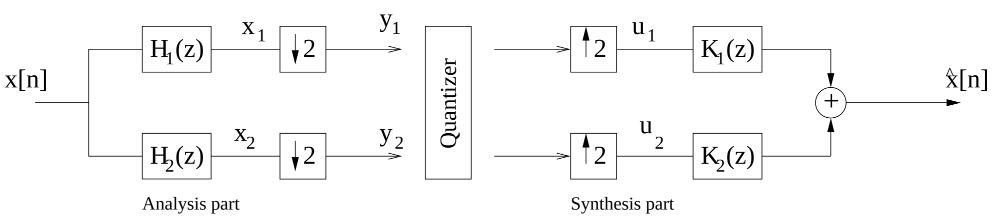

Figure 1: The structure of a two-channel filter bank.

2 Review of Signals and Systems

Discrete-time systems have different types of system properties such as invertability, memory property, causality, stability, time invariance and linearity. Linearity and time invariance play a fundamental role in signal and system analysis because many physical processes can be modeled as a Linear Time Invariant (LTI) system. One of the primary properties of LTI systems is that any such system possesses the superposition property which means that if the input to an LTI system consists of a linear combination of signals

$$x[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots$$

then by superposition the output is given by

$$y[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \ldots$$

where $y_k[n]$ is the response to $x_k[n], k = 1, 2, 3, \ldots$ In general the input is represented with $x[n]$ and the output is represented with $y[n]$. Now the important point is that if we can represent the input to an LTI system in terms of a set of basic signals, we can then use superposition to compute the output of the system in terms of its responses to these basic signals. The following sub-section explains the characterization of discrete-time signals and LTI systems.

2.1 Discrete-Time LTI Systems: The Convolution Sum

In fact we can write any discrete-time signal as a shifted weighted sum of unit samples. That is

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{1}$$

where $\delta[n-k]$ is the shifted unit sample and given by

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Each discrete-time LTI system which is also called a "filter" is characterized by its impulse response. As the name implies impulse response of an LTI system is the response of the system to unit sample which is given by (2) for $k = 0$. It is represented as $h[n]$. Impulse response is nothing but a discrete sequence of numbers. One way of classifying filters is made according to their impulse responses. If the impulse response is a finite sequence then corresponding filter is called *Finite Impulse Response* (FIR) filter. Otherwise it is called *Infinite Impulse Response* (IIR) filter. Now let us denote the response of a discrete-time LTI system to shifted unit sample $\delta[n-k]$ as $h_k[n]$, then the response of the system to an arbitrary input $x[n]$ can be expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \tag{3}$$

According to (3) if we know the response of the system to the shifted unit samples, namely $h_k[n]$, then we can construct the response to an arbitrary input. If a linear system is also time-invariant then we have the following characterization

$$h_k[n] = h[n-k] \tag{4}$$

where $h[n-k]$ is the shifted impulse response. As a result, the output of the system to an arbitrary input can be found immediately for the LTI systems if the impulse response of the system is known.The output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] \tag{5}$$

which is called *the convolution sum.* Now we will move to two different methods which are also used to characterize an LTI system.

2.2 The Response of Discrete-Time LTI Systems to Complex Exponentials

Two other tools used to characterize an LTI system are the z-transform and the Discrete-Time Fourier Transform. Convolution sum approach gives the time domain characterization whereas these new tools give frequency domain characterization. Complex exponential sequences are eigenfunctions of discrete-time LTI systems. That is, suppose that an LTI system with impulse response $h[n]$ has as its input the signal

$$x[n] = z^n \tag{6}$$

where $z = re^{j\phi}$ is a complex number which has an amplitude r and a phase ϕ, j is $\sqrt{-1}$ in electronics terminology (because i is used for current). The output of the system can be determined from the convolution sum as

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} \tag{7}$$

From this expression we see that if the input is the complex exponential given by (6), then the output is the same complex exponential multiplied by a constant that depends on the value of z, that is,

$$y[n] = H(z)z^n \tag{8}$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} \tag{9}$$

Here $H(z)$ is the eigenvalue associated with the eigenfunction z^n. This is very similar to eigenvalue eigenvector relation for matrices.

More generally, we see that equation (8) together with the superposition property, implies that the representation of signals in terms of complex exponentials leads to a convenient expression for the output of an LTI system. Specifically, if the input to a discrete-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x[n] = \sum_k a_k z_k^n \tag{10}$$

Then the output $y[n]$ will be

$$y[n] = \sum_k a_k H(z_k) z_k^n \tag{11}$$

In other words, the output can also be represented as a linear combination of the same complex exponential signals, and each coefficient in this representation of the output is obtained as the product of the corresponding coefficient a_k of the input and the system's eigenvalue $H(z_k)$ associated with the eigenfunction z_k^n.

Equation (9) is known as the z-transform. If we replace z with e^{jw} where w is the frequency, i.e., only take z's on the unit circle on the complex plane then we get Discrete-Time Fourier Transform (DTFT). Mathematically, it is

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jwn}. \tag{12}$$

$H(e^{jw})$ $\Big($or $H(z)\Big)$ is called the transfer function of the system and it characterizes the frequency response of the system. The inverse DTFT is given by

$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(e^{jw})e^{jwn}dw. \tag{13}$$

This integral is taken over any interval with length 2π because DTFT is 2π periodic. Therefore only considering the DTFT of a signal in the interval $w \in [-\pi, \pi]$ is enough to learn the frequency content of that signal (or frequency response for the systems). In our problem we will concentrate on that interval only. If the frequency magnitude response, i.e., $|H(e^{jw})|$ of a system is high for $w \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and close to zero for $|w| \in [\frac{\pi}{2}, \pi]$ then that system is called a *low-pass filter* which means the systems passes the low frequency content of its input and attenuates the high frequencies. If the inverse is true then it is called a *high-pass filter*.

DTFT which is defined by (12) is also applicable to any signal and gives the frequency content of that signal. Therefore, we are able to represent any signal as a linear combination of complex exponentials. The following simple example explains the matter very well.

Example 1 *Write* $x[n] = \cos[w_1 n]$ *as a linear combination of complex exponentials?*

Solution: The Euler formula is

$$e^{\mp jw} = \cos[wn] \mp j\sin[wn].$$

Using that formula we can write $\cos[w_1 n]$ as a sum of complex exponentials

$$\cos[w_1 n] = \frac{1}{2}\Big(e^{jw_1} + e^{-jw_1}\Big) = \frac{1}{2}e^{jw_1} + \frac{1}{2}e^{-jw_1}.$$

Figure shows the DTFT of $\cos(w_1 n)$.

As seen on Figure 2, DTFT of cosine is impulses at $w = \pm(w_1 + 2\pi)$. Therefore, we can conclude that any signal that carries single frequency has an impulse in frequency domain corresponding to that frequency, so any signal that does not carry single frequency can be written in terms of those signal which carry single frequency. This simply means that any signal can be written as a sum of weighted sum of sines and cosines. In example 1 we saw that cosine can be written in terms of complex exponentials so any signal can be written in the same form.

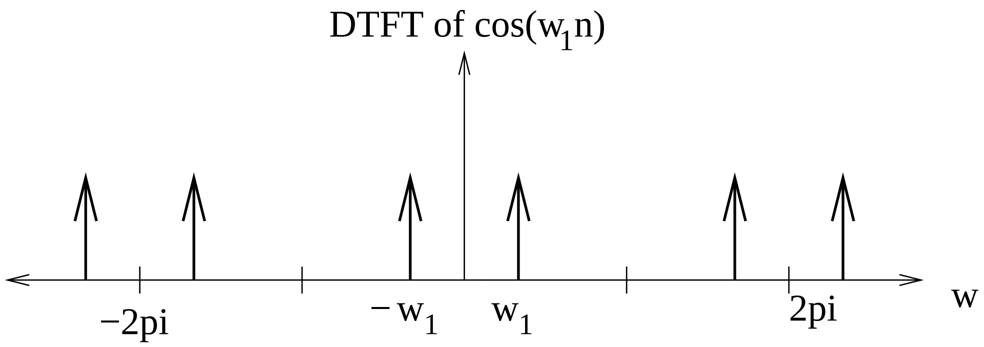

Figure 2: DTFT of $\cos(w_1 n)$.

To sum up, there are three different tools for examining discrete-time LTI systems. One for time-domain analysis and two for the frequency-domain. Time domain method is the convolution sum and it relates the input and the output as

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k].$$

The z-transform and DTFT relates the input and the output as

$$y[n] = \sum_{k} a_k H(z_k) z_k^n.$$

If we write everything in the frequency-domain then we have

$$Y(z) = H(z)X(z)$$

where $X(z)$ is the z-transform of the input. Therefore we conclude that convolution in time-domain corresponds to multiplication in frequency-domain. This is the end of the review of signals and systems. Now we will continue with the mathematical formulation of the two-channel filter bank.

3 Perfect Reconstruction (PR) Condition for a Two-Channel Filter Bank

The structure of a two-channel filter bank is going to be given again by Figure 3 for convenience.

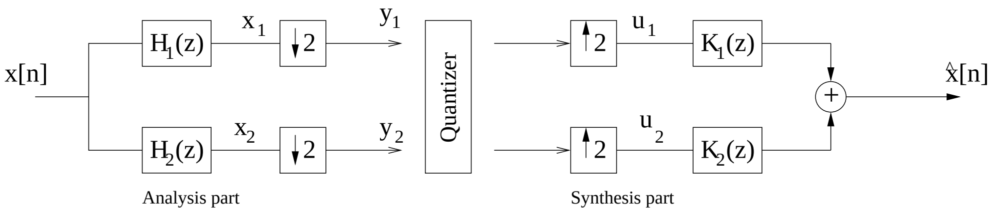

Figure 3: The structure of a two-channel filter bank.

In this structure $\downarrow 2$ represents a downsampler or a decimator. It simply keeps even-indexed samples of its input and discards the odd-indexed ones so in time-domain for the upper downsampler we have

$$y_1[n] = x_1[2n].$$

Frequency-domain equivalent of that expression is

$$Y_1(z) = \frac{1}{2}\Big(X_1(\sqrt{z}) + X_1(-\sqrt{z})\Big).$$

After downsampler the number of samples are reduced by half. $\uparrow 2$ represents an upsampler or an expander. It puts a zero between each samples of its input so in time-domain for the upper upsampler when there is no quantizer we have

$$u_1[n] = \begin{cases} y_1[\frac{n}{2}] & \text{n is even} \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

In frequency-domain characterization of upsampler is

$$U_1(z) = Y_1(z^2).$$

Contrary to downsampler, upsampler increases the number of samples by 2. Using the tools explained in the previous section and the frequency-domain characterizations of upsampler and downsampler now we can write the following relation in frequency-domain for the input $X(z)$ and the output $\widehat{X}(z)$:

$$\widehat{X}(z) = \frac{1}{2}\Big[T(z)X(z) + S(z)X(-z)\Big] \tag{15}$$

$$T(z) = [H_1(z)K_1(z) + H_2(z)K_2(z)] \tag{16}$$

$$S(z) = [H_1(-z)K_1(z) + H_2(-z)K_2(z)] \tag{17}$$

Perfect reconstruction (PR) is achieved when there is no quantizer if, in frequency domain,

$$\widehat{X}(z) = z^{-n_0} X(z) \tag{18}$$

or in time-domain

$$\hat{x}(z) = x[n - n_0] \tag{19}$$

is satisfied. z^{-n_0} corresponds to n_0 sample delay in time-domain. Therefore the output $\hat{x}[n]$ is the n_0 samples shifted version of the input $x[n]$. Mathematically this requires $T(z) = 2z^{-n_0}$ and $S(z) = 0$. There are different choices that satisfy this requirement but an important one was made by Smith and Barnwell [3] which results in Quadrature Mirror Filters (QMF). Therefore this choice gives QMF banks. The selection is made as follows. First we decide the low-pass filter $H_1(z)$ then the high-pass filter $H_2(z)$ is given by

$$H_2(z) = -z^{-N} H_1(-z^{-1}).$$

where N is the biggest index which corresponds to a non-zero sample in $h_1[n]$ (also called as the degree of the filter) and is used to make $H_2(z)$ causal. N must be an odd integer due to some frequency domain restrictions such as amplitude degradation at frequency $w = \frac{\pi}{2}$. Since $H_1(z)$ is a low-pass filter and $H_2(z)$ is high-pass, the low-frequency content of the input is carried by the upper branch and the high-frequency content is carried by the lower branch. The synthesis filters $K_1(z)$ and $K_2(z)$ are selected according to

$$K_1(z) = H_2(-z), \quad K_2(z) = -H_1(-z).$$

This selection gives a low-pass filter $K_1(z)$ and a high-pass filter $K_2(z)$. Now we can write everything in terms of $H_1(z)$ so

$$\widehat{X}(z) = \frac{1}{2} z^{-N} \Big[H_1(z) H_1(z^{-1}) + H_1(-z^{-1}) H_1(-z) \Big] X(z) = 2z^{-n_0} X(z).$$

Thus we have

$$z^{-N} \Big[H_1(z) H_1(z^{-1}) + H_1(-z^{-1}) H_1(-z) \Big] = 2z^{-n_0} \tag{20}$$

Minimum delay occurs if $n_0 = N$. Thus let us choose it like that. The final form form of (20) is

$$H_1(z) H_1(z^{-1}) + H_1(-z^{-1}) H_1(-z) = 2.$$

After defining the product filter $P(z) := H_1(z)H_1(z^{-1})$ we get

$$P(z) + P(-z) = 2. \tag{21}$$

(21) defines a half-band filter $P(z)$. $P(z)$ has $2N + 1$ coefficients where N is the order of $H_1(z)$ which was defined before and coefficient of $P(z)$ are symmetric with respect to origin. In (21) on the left hand side there are only even powers of z. Therefore, the right hand side requires that only the constant term which corresponds to z^0 is 1 and all other coefficients corresponding to z^{2n} must be zero. Thus we can write the following for $p[n]$

$$p[n] = \sum_{k=n}^{N} h_1[k]h_1[k-n] \tag{22}$$

$$p[2n] = \begin{cases} 2 & \text{n=0} \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

which means even-indexed coefficients of $p[n]$ are zero except the zeroth coefficient. (22) is the time domain equivalent of $P(z) = H_1(z)H_1(z^{-1})$ and $p[n]$ is the impulse response of the half-band filter $P(z)$. Then $P(z)$ is characterized by the set of $\frac{N+1}{2}$ coefficients p_{2n-1}'s which are odd-indexed coefficients of its impulse response $p[n]$. Thus

$$P(z) = 1 + \sum_{n=1}^{(N+1)/2} p_{2n-1}(z^{-(2n-1)} + z^{2n-1}) \tag{24}$$

The design of a PR-QMF bank starts with finding a half-band filter $P(z)$ first and then $H_1(z)$ is determined by spectral factorization of $P(z) = H_1(z)H_1(z^{-1})$.

4 LP formulation

4.1 Verbal Description of the Model

In section 3 it was shown that it is possible to obtain a two-channel PR-QMF bank by expressing all four filters in terms of a prototype low-pass filter. PR can be achieved if there is no quantizer between analysis and synthesis part. A quantizer is a device which converts its input into finite number of possible outputs. This is essential otherwise we need infinite number of bits to represent the outputs of the analysis part. Input output relation for a quantizer is shown on Figure 4.

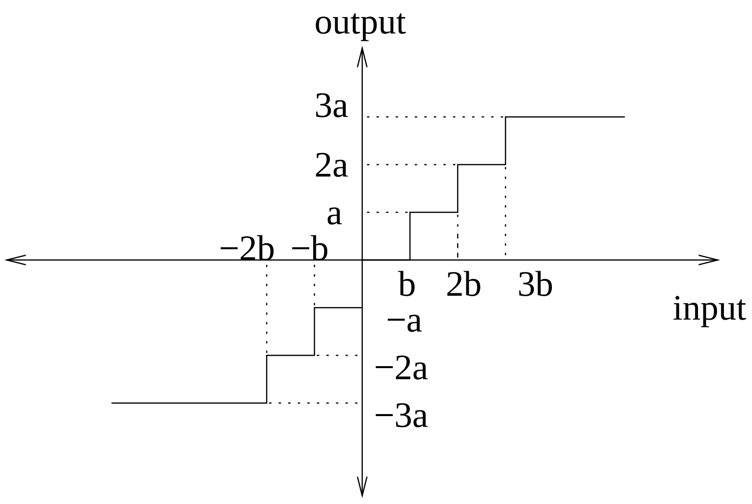

Figure 4: Input-output relation of a uniform quantizer.

Hopefully error created by quantization cannot be sensed by our ears and eyes if a proper design is made. In real life the analysis part is at some place and the synthesis part is at some other place. In between we have data transmission, i.e., analysis part is the transmitter and the synthesis part is the receiver. As an another real life application we process a signal and compress the outputs of analysis part and store the output in somewhere. Then the synthesis part is used to decompress the signal. In such applications in order to reduce the bit rate we need to code subband signals $y_1[n]$ and $y_2[n]$. Example 2 demonstrates how reduction of bit rate is achieved using subband coding.

Example 2 *Assume that we have a signal sampled at 10kHz, i.e., there are 10000 samples for a duration of 1 sec. Further assume that each sample of that signal is represented with 16 bits so data rate is 160 kbits/sec. Most of the real life signals are low-pass in nature, i.e., most of the energy of that signal is concentrated at lower frequencies (for $w \leq \frac{\pi}{2}$). Then subband coding provides the following. First decompose the signal into subbands and then treat them separately. After downsampling magnitudes of input samples are decreased so now we can represent them using fewer bits. This is similar to the fact that for binary representation of 20 we need 5 bits (25=11001), for 10 we need 4 bits (10=1010). Now $y_2[n]$ carries very small amount of energy with respect to $y_1[n]$ because it carries the high frequency content and there very small amount of energy in high frequency bands. Assume that each* *sample of $y_1[n]$ is represented with 8 bits and that of $y_2[n]$ is represented with 3 bits. Due to downsamplers sampling rate is reduced to 5kHz. Thus new data rate is*

$$\textit{data rate}=8*5+3*5=55 \textit{ kbits/sec}$$

As a result we reduced the data rate from 160 kbits/sec to 55 kbits/sec.

In this problem the objective is to maximize the coding gain which is explained in more detail in the following subsection.

4.2 Mathematical Formulation of Primal Problem

4.2.1 Formulation of Objective Function

A standard objective measure of the compression capability of orthogonal subband coding is the coding gain

$$G_{SBC} = \frac{D_{PCM}(R)}{D_{SBC}(R)} \tag{25}$$

expressing the ratio of squared-error distortions for pulse-code modulation (PCM) and subband coding (SBC) for a given bit rate R (PCM is a very primitive type of coding). Orthogonality arises due to selection of filters as QMF and it means that $h_1[n]$ is orthogonal to $h_2[n]$, i.e,

$$\sum_n h_1[n]h_2[n-2k] = 0 \quad \forall k \quad (k \text{ is an integer}).$$

Assuming high-rate quantization and optimal bit allocation between the subbands, the coding gain for the 2-band (or channel) system becomes

$$G_{SBC} = \frac{\left(\frac{1}{2}\right)\sum_{m=1}^{2}\sigma_m^2}{\left(\prod_{m=1}^{2}\sigma_m^2\right)^{1/2}} = \frac{\frac{1}{2}(\sigma_1^2+\sigma_2^2)}{(\sigma_1^2\sigma_2^2)^{1/2}} \tag{26}$$

where σ_m^2 is the variance of the signal in band m. Thus σ_1^2 is the variance of y_1 and σ_2^2 is that of y_2 in Figure 3. Since the objective is related to analysis part we will concentrate on that part only and will ignore the synthesis part. Due to orthogonality condition, $\sigma_1^2 + \sigma_2^2$ is equal to the variance of the input signal $x[n]$. It follows that the maximum coding gain is attained by maximizing $\xi(h_1) := \frac{1}{2}\sigma_1^2$ over the impulse response $h_1[n]$ of the low-pass filter $H_1(z)$ which is the prototype filter. Normalization factor $\frac{1}{2}$ is used for convenience. If we write (26) as sum of fractions

$$G_{SBC} = \frac{\frac{1}{2}\sigma_1}{\sigma_2} + \frac{\frac{1}{2}\sigma_2}{\sigma_1} \tag{27}$$

we see that maximizing $\frac{1}{2}\sigma_1^2$ means that the energy in the first channel, i.e, energy of $y_1[n]$ will be maximized (energy of a signal is equal to the difference between its variance and the square of its mean, therefore energy is directly proportional to variance). Thus $y_2[n]$ will have very small amount of energy which reduces the bits required to code $y_2[n]$. Explicit form of $\xi(h_1)$ is

$$\xi(h_1) = \frac{1}{2}h_1^T\mathbf{R}\mathrm{h}_1 \tag{28}$$

where $\mathbf{R}$ is the $(N+1)$x$(N+1)$ covariance matrix of the input. Covariance matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \mathrm{E}[\mathrm{x} * \mathrm{x}^{\mathrm{T}}] - \mu * \mu^{\mathrm{T}} \tag{29}$$

where μ is the mean vector of the input $x[n]$. Here we assume that the input signal $x[n]$ is *wide sense stationary* with Toeplitz property. Then in matrix notation $\mathbf{R}$ becomes

$$\mathbf{R} = \begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_N \\ r_1 & r_0 & r_1 & \cdots & r_{N-1} \\ r_2 & r_1 & r_0 & \cdots & r_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_N & r_{N-1} & r_{N-2} & \cdots & r_0 \end{bmatrix}$$

$\mathbf{R}$ satisfies symmetry property in addition to Toeplitz property , so evaluation of $\xi(h_1)$ given by (28) has a simple format. That simple format is characterized by the odd-indexed coefficients p_{2n-1} of the half band filter

$P(z)$ given by (24). Now let us drive it.

$$
\begin{aligned}
\xi(h_1) &= \frac{1}{2} h_1^T \mathbf{R} \mathrm{h}_1 \\
\xi(h_1) &= \begin{bmatrix} h_1[0] & h_1[1] & \cdots & h_1[N] \end{bmatrix} \begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_N \\ r_1 & r_0 & r_1 & \cdots & r_{N-1} \\ r_2 & r_1 & r_0 & \cdots & r_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_N & r_{N-1} & r_{N-2} & \cdots & r_0 \end{bmatrix} \begin{bmatrix} h_1[0] \\ h_1[1] \\ \cdots \\ h_1[N] \end{bmatrix} \\
\xi(h_1) &= \frac{1}{2}\Big[r_0 \sum_{n=0}^{N} h_1^2[n] + 2r_1 \sum_{n=1}^{N} h_1[n]h_1[n-1] + \cdots + 2r_N \sum_{n=N}^{N} h_1[n]h_1[n-N] \Big] \\
\xi(h_1) &= \frac{1}{2} r_0 \sum_{n=0}^{N} h_1^2[n] + \sum_{k=1}^{N} r_k \sum_{n=k}^{N} h_1[n]h_1[n-k] \\
\xi(p_k) &= \frac{1}{2} r_0 p_0 + \sum_{k=1}^{N} r_k p_k
\end{aligned}
\tag{30}
$$

where p_k is defined by (22) (here we used subscript instead of indices for convenience). p_{2k}'s are zero except for $k = 0$ by (23). Thus we can write the objective function ξ as

$$
\xi(p_{2k-1}) = \frac{1}{2} r_0 p_0 + \sum_{k=1}^{(N+1)/2} r_{2k-1} p_{2k-1} = \frac{r_0}{2} + \sum_{k=1}^{(N+1)/2} r_{2k-1} p_{2k-1} \tag{31}
$$

where $p_0 = 1$ comes from the half-band requirement. Now we found an explicit expression for the objective function which is linear in p_{2k-1}'s.

4.2.2 Formulation of Constraints

Constraints come from the half-band filter $P(z)$. For convenience expression for $P(z)$ is repeated here.

$$
P(z) = H_1(z) H_1(z^{-1}) \tag{32}
$$

If we move from the z-transform to DTFT we get

$$
P(e^{jw}) = H_1(e^{jw}) H_1(e^{-jw}) = |H_1(e^{jw})|^2 \geq 0 \tag{33}
$$

The explicit form of $P(z)$ is given by (24). In terms of DTFT after using Euler formula, (24) is equivalent to

$$\begin{aligned} P(e^{jw}) &= 1 + \sum_{n=1}^{(N+1)/2} p_{2n-1}(e^{-jw(2n-1)} + e^{jw(2n-1)}) \\ &= 1 + 2\sum_{n=1}^{(N+1)/2} p_{2n-1}\cos\Big(w(2n-1)\Big) \geq 0 \end{aligned} \tag{34}$$

for $0 \leq w \leq \pi$. This is the only constraint for the problem.

Here there are finite number of parameters but the number of constraints is infinite because w can take infinitely many values. Therefore this type of problems are called *linear semi-infinite programming (SIP) problems.*

In conclusion primal SIP problem is

$$\begin{array}{rc} max & \xi(p_{2k-1}) = \frac{r_0}{2} + \sum_{k=1}^{(N+1)/2} r_{2k-1}p_{2k-1} \\ subject\ to & 1 + 2\sum_{n=1}^{(N+1)/2} p_{2n-1}\cos\Big(w(2n-1)\Big) \geq 0 \\ & 0 \leq w \leq \pi \end{array}$$

4.3 Dual SIP Problem

Dual methods in general are based on the viewpoint that the Lagrange multipliers are the fundamental unknowns associated with constrained optimization problems. Once these multipliers are known, the solution may often be easily determined. In fact Lagrange multipliers are the variables of the dual problem. The dual of an elementary LP with N variables and M inequality constraints is an optimization problem with M variables (the Lagrange multipliers) and N equality constraints. However, primal SIP (PSIP) has infinitely many constraints, so an extension of elementary LP is needed. The semi-infinite dual problem DSIP has as its variables generalized finite sequences $\lambda(w)$. First let's rearrange PSIP into a form for which it is easy to take the dual.

(PSIP)

$$\begin{array}{rc} max & \xi(p_{2k-1}) = \frac{r_0}{2} + \sum_{k=1}^{(N+1)/2} r_{2k-1}p_{2k-1} \\ subject\ to & -2\sum_{n=1}^{(N+1)/2} p_{2n-1}\cos\Big(w(2n-1)\Big) \leq 1 \\ & 0 \leq w \leq \pi \end{array}$$

(DSIP)

$$\begin{aligned} & min \qquad \vartheta(\lambda) = \tfrac{r_0}{2} + \textstyle\sum_w \lambda(w) \\ \textit{subject to} \qquad & -2\textstyle\sum_w \lambda(w)\cos\Big(w(2n-1)\Big) = r_{2n-1}, \qquad n = 1, \dots, \tfrac{N+1}{2} \\ & \lambda(w) \geq 0 \\ & 0 \leq w \leq \pi \end{aligned}$$

$\frac{r_0}{2}$ is kept in both objective functions for convenience since it does not have any effect other than shifting the objective values. In fact, in the constraints of DSIP there must be integration instead of sum because w is a continuous parameter. However summation is used because there are finite non-zero λ's but we have to consider all possible choices in the interval $0 \leq w \leq \pi$.

Table 1 summarizes the variables and parameters used in PSIP and table 2 does the same thing for DSIP.

Table 1: Variables and parameters used in PSIP.

p_{2n-1}	variable in PSIP, n runs from 1 to $\frac{N+1}{2}$,
ξ	objective function,
w	frequency, continuous parameter, infinitely many values,
r_{2n-1}	parameter, first row of covariance matrix of the input.

Table 2: Variables and parameters used in DSIP.

$\lambda(w)$	variable in DSIP, there are infinitely many but only finite number of them are nonzero,
ϑ	objective function,
w	frequency, continuous parameter, infinitely many values,
r_{2n-1}	parameter, first row of covariance matrix of the input.

4.4 Characterization of Optimal Filters

The analysis of DSIP leads to a characterization of the optimal filters in terms of theirs roots (called zeroes in EE terminology) on the unit circle. The duality gap is defined by the difference between objective values of dual and primal problems. Mathematically, it is

$$
\begin{aligned}
Duality \quad gap &= \vartheta(\lambda) - \xi(p_{2k-1}) \\
&= \sum_w \lambda(w) - \sum_{k=1}^{(N+1)/2} r_{2k-1} p_{2k-1} \\
&= \sum_w \lambda(w) - \sum_{k=1}^{(N+1)/2} p_{2k-1} \Big\{ -2 \sum_w \lambda(w) \cos\Big((2k-1)w\Big) \Big\} \qquad (35) \\
&= \sum_w \lambda(w) \Big\{ 1 + 2 \sum_{k=1}^{(N+1)/2} p_{2k-1} \cos\Big((2k-1)w\Big) \Big\} \\
&= \sum_w \lambda(w) P(e^{jw})
\end{aligned}
$$

The duality gap is zero for the optimal solutions p^*_{2k-1} and $\lambda^*(w)$. Now assume that $\exists\ k$ such that $r_{2k-1} \neq 0$. Then there exist an optimal filter $P(e^{jw})$, an integer $K \in [1, \ldots, \frac{N+1}{2}]$, and certain constants $\lambda^*(w_k)$ and w_k such that $\lambda^*(w_k) > 0$, $w_k \neq w_l$ if $k \neq l$. Then following equation can be written.

$$
r_{2n-1} = -2 \sum_{k=1}^{K} \lambda^*(w_k) \cos\Big(w_k(2n-1)\Big) \quad 0 \leq n \leq \frac{N+1}{2} \qquad (36)
$$

In fact, (36) is nothing but the constraint of DSIP. Therefore, $\lambda^*(w_k)$'s are the feasible solutions to the dual problem. Addition to (36) if

$$
P(e^{jw_k}) = 0, \quad 1 \leq k \leq K \qquad (37)
$$

holds then complementary slackness is satisfied and $\lambda^*(w_k)$ is the dual optimal solution. Because by (35), complementary slackness requires

$$
\sum_w \lambda(w) P(e^{jw}) = 0,
$$

by (36) $\lambda^*(w_k) > 0$ so the only way to satisfy complementary slackness is to satisfy (37). Therefore, (36) and (37) are necessary and sufficient for optimality. Now why we choose $K \in [1, \ldots, \frac{N+1}{2}]$ is more clear because there can be at most $\frac{N+1}{2}$ nonzero $\lambda^*(w_k)$ all other $\lambda^*(w)$'s at the optimum point are zero again by complementary slackness. This also clarifies the matter why *sum* is used instead of *integral* in the constraints of DSIP. As a result, the set of optimal filters are characterized by the zeroes on the unit circle and those zeroes are w_k's. Physically, a zero is a certain frequency. If the frequency of the input is same as the zero of the system then no matter how big the input is the output of the system is zero.

5 An Algorithm to Solve SIP Problem: The Discretization Method

A simple and intuitive idea for solving the SIP problem is based on discretization of the frequency axis, i.e., discretization of w. By this way the number of constraints are made finite and such a problem can be solved using standard simplex method. However, the solution of discretized problem is just an approximation to the original SIP problem. Now discretized version is
(discretized PSIP)

$$\begin{array}{ll} & max \quad \xi(\tilde{p}_{2k-1}) = \frac{r_0}{2} + \sum_{k=1}^{(N+1)/2} r_{2k-1}\tilde{p}_{2k-1} \\ subject\ to & -2\sum_{n=1}^{(N+1)/2} \tilde{p}_{2n-1}\cos\Big((2n-1)w_k\Big) \leq 1 \\ & 0 \leq w_k \leq \pi \qquad w_k = \frac{\pi}{M}k \qquad k = 0, 1, \ldots, M \end{array}$$

(discretized DSIP)

$$\begin{array}{ll} & min \quad \vartheta(\tilde{\lambda}) = \frac{r_0}{2} + \sum_{k=0}^{M} \tilde{\lambda}(w_k) \\ subject\ to & -2\sum_{k=0}^{M} \tilde{\lambda}(w_k)\cos\Big((2n-1)w_k\Big) = r_{2n-1}, \qquad n = 1, 2, \ldots, \frac{N+1}{2} \\ & \tilde{\lambda}(w_k) \geq 0 \\ & 0 \leq w_k \leq \pi \end{array}$$

In general the optimal solution $\tilde{p}^*_{2k-1}$ for the discretized problem is not feasible for the PSIP problem. Following algorithm is used to find a feasible solution p^*_{2k-1} to PSIP problem using the optimal solution $\tilde{p}^*_{2k-1}$ of the discretized PSIP problem. The summary of algorithm is

1. Define $w_k = \frac{\pi}{M}k \qquad k = 0, 1, \ldots, M$
   solve discretized DSIP using simplex algorithm,
2. Determine w_k's corresponding to nonzero $\tilde{\lambda}$'s
3. Compute coefficients $\hat{p}_{2k-1}$ to satisfy $P(e^{jw_k}) = 0$
4. Find the minimum δ of $P(e^{jw})$ for computed coefficients $\hat{p}_{2k-1}$
5. Let $p^*_{2k-1} = \frac{\hat{p}_{2k-1}}{1+\delta}$
6. Compute the half-band filter $P(e^{jw})$ defined by p^*_{2k-1}

7. Compute spectral factors of $P(e^{jw}) = H_1(z)H_1(z^{-1})$.

The solution generated by this algorithm satisfies the feasibility condition for the original PSIP problem. For $M \approx 10N$ discretization method works well [4].

6 Conclusion

We have shown that designing an orthogonal QMF bank to maximize the coding gain by LP is possible. It appeared that the coding gain is a linear function of the odd coefficients of the half band filter $P(z)$. After formulation was done we saw that the problem in our hand was a maximization type linear SIP problem. since there are infinitely many number of constraints we discretized the SIP problem. After discretization SIP turned out to be an elementary LP which can be solved by simplex algorithm. However, here the problem was that the solution of the discretized problem was infeasible for the SIP problem. We proposed an algorithm to make it feasible for the SIP problem. If the sampling in the frequency domain is high enough, i.e., $M \approx 10N$, then the results are quite well. We also obtained the dual problem. The dual problem provided an upper bound for the primal one. In addition to that the dual also provided a useful information about the places of zeroes of the optimal half-band filters. Those optimal filters are characterized by their zeroes on the unit circle in the complex domain. Finally the analysis and synthesis filters are evaluated from spectral factorization of the half-band filter $P(z) = H_1(z)H_1(z^{-1})$. The only limitation of that model is the assumption made about the statistics of the input signal. The input was assumed to be wide sense stationary and whose covariance matrix was Toeplitz.

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Linear, Non-approximated Models for Optimal Routing in Hazardous Environments

by A. Őzçevik and G. Çelik

1 Verbal Description of the Model

This model is a linear optimization for the routing of vehicles through hazardous environments or for routing vehicles carrying hazardous materials.The model seeks to find the route with the minimum cost and the minimum probability of accident.Finding the best route for a vehicle in a hazardous environment means finding a route with a low probability of accident.The goal is to protect the vehicle from the environment,or from some specific danger,which may depend on the route taken by it. Finding the best route for a hazardous material(HAZMAT)is a related problem,in which the environment is to be protected from the vehicle specially if an accident happens in which HAZMAT is spilled,or released in some fashion.In both cases the route is better if the cost of routing the vehicle through it's low. In this model in order to develop and expression of the probability of accident on each potential path between an origin and a destination we have to know the probability of an accident on each arc of the network.Then we can look for the path with the minimal probability of accident.If p_{ij} is the probability of accident on arc (i,j),the total probability of accident on a path is

$$\sum_{(i,j)\in path} \Big(\prod_{(k,l)<(i,j)} (1-p_{kl})p_{ij} \Big)$$

where $(k,l) < (i,j)$means that the arc (k,l)is previous to (i,j) on the path.The summation is over all the arcs on the path.Each term in the summation is the probability of accident on that particular arc of the path,given that the vehicle has had no accident on all previous arcs on the path.In order to use this expression in a mathematical programming formulation ,each probability has to be multiplied by a zero -one variable,which takes the value one in the solution if the corresponding arc on the path, and zero if it is not.The resulting expression is nonlinear.If the probability is p_{ij} are assumed very small ,the terms (1-p_{ij}) are approximately one ,and the expression reduces to

$$\sum_{(i,j)\in path} p_{ij}$$

which leads to a linear expression, when the probabilities are multiplied by the zero-one arc choice variables.

This form is appropriate when individual probabilities are small.If the probability of not having an accident on an arc is used instead of the probability of having an accident, then we get a new expression ,which is applicable at all levels of accident probabilities,that can be written for the probability of an accident-free trip.This new expression can be easily linearized. On each arc of the road network, and for a particular shipment of materials ,there are known values of probabilities of accidents in categories 1 and 2(in category 1,as a consequence of an accident, neither human lives are lost nor wild life or environment are threatened and then the accident is perceived as without much importance.If there is risk to the wild life or environment or there are any human lives involved ,the accident is perceived as a very bad one and we place the accident in category2). In order to compute these probabilities the concept of risk profile is used.The risk profile is constructed by plotting 1-F(x) vs x, where x is some measure of the consequences of an accident (increasing values of x mean increasing consequences of the accident), and F(x) is the cumulative distribution function of accidents.The shape of the risk profile depends on several conditions: first ,it will be different for hazardous materials and for hazardous environments.It will also depend on the type of hazardous material ,on the type of carrying vehicle, on the geographical and meteorological conditions, on the number of people living in the area and so on.

Once the risk profile is plotted, the area under the curve will represent the total probability of accidents on that arc of the network.This probability is independent of the probabilities of accidents on other arcs.Also ,a value of x(x*) is selected ,above which,all accidents are of category2.The area under the curve and above x*,represent the probability of accidents of category2. In this model it is used the probability of an environment-threatening accident(category 2)occurring.If such an accident happens the cost of it will be C ,and the aim is to keep the probability of such a severe accident happening on any place of the route,smaller than a value of α .Alternatively this may be the probability of an accident in a hazardous environment.

2 Mathematical Description of the Model

In this model we will consider two cases one is the one- shipment problem and the other one is multiple origins and multiple destinations. The one shipment problem: Let G be a road network consisting of N nodes and A links or arcs .Let R_{od} be the set of all possible routes or paths from an origin node o to a destination node d. For each arc(i,j)$\in$ A ,there is a probability p_{ij} of an accident of category 2.We will call the probability of of the shipment not having an accident of category 2 the probability of survival on arc (i,j), q_{ij}. Then the probability of the survival on a particular route r $\in$ R_{od} is

$$\prod_{(i,j)\in r} q_{ij}$$

If we define the variable x_{ij} as having a value of one if the arc (i,j) is on the path r , and zero otherwise the survival probability on any path r in A that is chosen is also :

$$\prod_{(i,j)\in A} q_{ij}^{x_{ij}}$$

That is, the probability of an accident free journey (no accident in category 2) is the product of the probabilities of no accident on any of the arcs chosen for the path.We may want to constrain this probability of a no-accident journey to be better or at least or equal to present value α,that is $\prod_{(i,j)\in A} q_{ij}^{x_{ij}} \geq \alpha$. This equation is nonlinear ,but it can be easily linearized by taking the logarithm of both sides

$$\sum_{(i,j)\in A} x_{ij}|logq_{ij}| \leq |\log \alpha|$$

If c_{ij} is the cost of routing trough arc (i,j) , the resulting model that minimizes transport cost subject to a constraint on the probability of an accident-free trip can be written as

P1

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{1}$$

subject to

$$\sum_{j\in O_o} x_{oj} = 1 \tag{2}$$

$$\sum_{i\in l_d} x_{id} = 1 \tag{3}$$

$$\sum_{i\in l_k} x_{ik} = \sum_{j\in O_k} x_{kj} \qquad \forall k \in N, k \neq o, d \tag{4}$$

$$\sum_{(i,j)\in A} x_{ij}|logq_{ij}| \leq |log\alpha| \qquad (5)$$

and as defined before , x_{ij} =0 or 1 $\forall$ (i,j) in A . In these equations o and d are the origin and destination nodes, respectively ;O_o is the set of nodes j that there exists a direction arc (o,j); I_d is the set of nodes j such that there exists a directed arc (j,d) ; O_k is the set of nodes j such that there exists a directed arc (k,j) ,and l_k is the set of nodes j such that there exists a directed arc (j,k).Note that the solution could be infeasible if α is too high and the survival probabilities on each r too low. However we can solve the problem as a multi objective problem by bringing constraint (5) to the objective function as an objective to be minimized. The relaxed problem is always feasible and is the well studied bicriterion shortest path problem. Thus,shortest path algorithms can be used. The weighting method is the appropriate multiobjective approach for this problem, because it preservers the unimodular constraint matrix if LP is to be used , or it preserves the ability to apply one of the standard shortest path algorithms. The problem using the weighting method is stated as :

$$min\omega \sum_{(i,j)\in A} c_{ij}x_{ij} + (1-\omega) \sum_{(i,j)\in A} x_{ij}|logq_{ij}| \quad 0 \leq \omega \leq 1 \quad (6)$$

Subject to constraints (2) ,(3) and (4).If we rewrite this objective as

$$min \sum_{(i,j)\in A} [\omega c_{ij} + (1-\omega)|logq_{ij}|]x_{ij}$$

the model can be seen as shortest path problem with arc lengths $[\omega c_{ij} + (1-\omega)|logq_{ij}|]$,and solved using a shortest path algorithm by linear programming. For each value of ω applied ,a value of cost and probability of an accident-free journey is generated. As ω is varied from large to small , the path chosen shifts from a lower cost , higher risk one to a higher cost , lower risk path.The probability of an accident-free journey , of course ,is calculated outside the model using the product of the probability of arc survivals on the route chosen. That is ,the probability is

$$\prod_{(i,j)\in A} q_{ij}^{x^*_{ij}}$$

where $x*_{ij}$ indicate the arcs chosen in the optimal solution.

Multiple origins and multiple destinations(or multiple shipments) model, with constrained risk on each arc: If there are multiple origins and multiple destinations, and total risk is constrained or optimized ,as in the previous case , the problem can be decomposed into as many problems as there are origin-destination (O-D) pairs. For each of these pairs ,there is an optimum path. In this case ,different paths, corresponding to different O-D pairs, are very likely to share some low-risk arcs. Similarly ,if multiple shipments are to be routed between a single origin and a single destination , the same optimal path must be used every time . However , if equity is an issue , the risk on each arc should not exceed a predetermined value (either system-wide, or arc-specific).In these cases , either for multiple O-Ds or for multiple shipments as more shipments pass through a particular arc ,the probability of an accident on that arc increases. Thus ,the problem is no longer separable , it can be cast as the shipment of multiple commodities.For multiple O-Ds we define the variable x_{ij}^{kl}as one ,if the arc (i,j)belongs to the optimal path for origin-destination pair (k,l)$\in$ OD(the set of origin destination pairs), and zero otherwise.We also define q_{ij}^{kl} as the probability of survival of a shipment going from k to l , on arc (i,j) . Several measures of risk can be considered:

a) Overall risk of accidents (risk of an accident on the network)

b) Risk for each OD pair

c) Risk on each arc due to all shipments.

The overall risk constraint is

$$\prod_{(i,j)\in A,(k,l)\in OD} (q_{ij}^{kl})^{x_{ij}^{kl}} \geq \alpha \qquad (7)$$

This constraint forces the joint probability of not having accidents on any arc (due to any shipment) to be greater than or equal to a preset value(this is equivalent to forcing the probability of accident to be at most 1-α).Also,for each OD pair ,a risk constraint can be written :

$$\prod_{(i,j)\in A} (q_{ij}^{kl})^{x_{ij}^{kl}} \geq \alpha_{kl} \qquad \forall (k,l) \in OD \qquad (8)$$

Note that the constraint (7)is an aggregated version of constraint (8) if

$$\alpha = \prod_{(k,l)\in OD} \alpha_{kl}$$

Finally a constraint may be written on the probability of no accident on the particular arc (i,j) due to all possible paths or shipments that go through it. Again ,using the probability that no accident happens on that arc, due to any OD pair of shipment ,the constraint is :

$$\prod_{(k,l)\in OD} (q_{ij}^{kl})^{x_{ij}^{kl}} \geq \alpha_{ij} \qquad \forall (i,j) \in A \qquad (9)$$

By taking their logarithms ,all of these constraints can be linearized and placed in the objective , individually or together, as we did in the one-origin -one-destination cases. If we bring all the linear versions of these constraints , or a subset of them to the objective , the problem can be cast as a multiobjective problem, taking now the form of a multiple-criterion, multiple origin-destination ,shortest path problem. We must note however, that in practice ,as we include more terms in the objective , the solution of the problem becomes more difficult if the weighted method is used ,and that is really the only practicable method for such problems as these ,either the linear programming problem is solved or a shortest path algorithm is applied to derive the needed integer solutions. It is in the adjustment of the weights to achieve specific levels of cost and of the probability of accident that the difficulty comes about.

Although there is no barrier to attempting to place the linearized versions of (7),(8),and (9) in the objective ,the last one alone would add as many objective terms as there are arcs in the network .Or, at least ,as many terms as arcs for which we want to upper bound the probability of accident; each one with a weight ω_{ij} . To circumvent this difficulty and to provide a better explanation , we state only the two objective problem in cost and the probability of accident in the system (the linearized left hand side of expression(7)): The multi-shipment problem;

$$P2 \qquad min\ Z_1 = \sum_{(i,j)\in A,(k,l)\in OD} c_{ij} x_{ij}^{kl} \qquad (cost) \tag{10}$$

$$Z_2 = \sum_{(i,j)\in A,(k,l)\in OD} (|log q_{ij}^{kl}|) x_{ij}^{kl} \qquad (probability\, of\ accident) \tag{11}$$

subject to

$$\sum_{j\in O_k} x_{kj}^{kl} = 1 \qquad \forall (k,l) \in OD \tag{12}$$

$$\sum_{i\in l_l} x_{ij}^{kl} = 1 \qquad \forall (k,l) \in OD \tag{13}$$

$$\sum_{i\in l_m} x_{im}^{kl} = \sum_{j\in O_m} x_{mj}^{kl} \qquad \forall m \in N, m \neq k, l \forall (k,l) \in OD \tag{14}$$

This formulation corresponds to a bicriterion , multiple-origin - multiple destination shortest path problem. For each choice of values of the weights on the objectives ,it could be solved using known O(n^3) algorithms for this problem. If the number of origins and destinations is small , as compared to the total number of nodes in the network ,it may be worth solving the problem by decomposition .This is done noting that the constraints of P2can be grouped in as many groups as there are origin -destination pairs. Also ,once a value has been chosen for the weights on the objectives ,the resulting objective can be expressed as the sum of as many terms as there are origin -destination pairs. Thus the problem separates naturally into as many sub problems as there are origin destination pairs, each sub problem having the form of problem P1 , with variables x_{ij}^{kl}.For the chosen value of w ,each sub problem is solved separately, using a 0(n^2)algorithm .For each value of ω ,the solutions are combined to form the original objectives. That is , the cost objectives of all the sub problems are added to form the original cost objective, and the probability objectives of all the sub problems are also ! added to obtain the original probability objective .The corresponding point in a cost/probability trade-off curve can be drawn. If a linearized version of (8) had been used to generate a set of additional objectives for problem P2 , this would add one extra objective to each sub-problem. If (9) in its linear version is also used , there will be as many new terms in each objective ,as arcs of interest. However ,the general procedures used for solving the problem remain the same .

3 Discussion

Most models that find the least -risk route for vehicle carrying hazardous materials ,also applicable to the finding of the least-accident routes in a hazardous environments, are non-linear .These models are linearized through approximation .These approximations are applicable when the risk is low,namely in the order of 0.0001% However, when the risk is much higher , as for example in cases of extremely bad weather conditions in a rural highway system or in military applications , they are not applicable ,because the approximation leads to large errors, and the models remain non-linear.By using the probability of survival on each arc of the route, we develop a model which is easily linearized, without any approximations .We use the probability of survival as an objective, since we believe that the usual expected consequences objective does not necessarily reflect people's perception. The proposed models can be viewed as bi or multiple criterion shortest path problems. Since shortest path problems have very well known polynomial solution methods ,the proposed models can be used in situations in which fast ,real time solution is important .Small computers have power enough for almost any real-size problems if these models are used. Furthermore , if a fixed cost can be somehow assigned a priori to the risk ,the problem becomes a one -objective problem, there is no need to use large sets of objectives. From a different point of view if a fixed value of α.(as in equation 7 to 9)is desired ,the Lagrangian Relaxation of problem can be solved, where equation (11) is the relaxed constraint .In this case ,for each weight on the relaxed constraint ,the problem can be decomposed into sub problems, as was done for the multiobjective weighting approach. Care must be taken ,though ,because some values of alpha could make the mathematical problem infeasible. From a practical point of view ,this would mean that the probabilities of accident on the arcs make it impossible to reach an overall probability of accident of 1-α.

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Optimal Capacity and Flow Assignment for Self-Healing ATM Networks Based on Line and End-to-End Restoration

by T. İlhan and D. Erşahin

1 Verbal Description of the Model

Living at a time named as "Information Age" no one can undervalue the importance of networks today. Increasing number of people carry out most of their interactions on an interrelated network. In correspondence with the networks' popularity, two qualities of the networks became crucial; speed and survivability. These are not independent from each other. Indeed, ceteris paribus, importance of networks' survivability can be defined as a function of its speed. In a high-speed network, even a short time of failure would result in loss of a large volume of data. Consequently, introduction of high capacity optical fiber was a breakthrough in terms of networks' efficiency since increasing speed made networks' reliability vital. Fast restoration from a network breakdown has become a key point for having more dependable systems in the emerging high-speed network environments.

To improve survivability, self-healing algorithms has been the most effective tool. Originally designed for synchronous transfer mode (STM) networks, self-healing algorithms became a successful introductory step for the development of virtual path (VP) restoration schemes, which accomplish rapid restoration in asynchronous transfer mode (ATM) networks.

Both self-healing algorithms and VP restoration schemes look for spare capacity and real time path rerouting in case of a network failure. Since idle capacity is used to reroute the affected traffic, the allocation of the spare capacity over the network when a failure happens becomes the key point for the success-quick restoration- of the schemes.

ATM networks fast restoration can be achieved by switching from failed virtual paths to others. There exist two different restoration strategies depending on the location where traffic rerouting has performed: line restoration and end-to-end restoration. In case of a failure, line restoration scheme provides alternative routes between two end nodes of the failed link- virtual path. On the other hand, end-to-end restoration scheme considers the whole network and finds alternative routes from the source to the destination node.

Although it has a much more complex rerouting process, end-to-end restoration scheme is widely preferred to line restoration scheme. This is due to end-to-end restoration scheme's effective use of the spare capacity leading to less capacity usage for restoration purposes. However, it should also be stated that this preference is just based on computer simulations and some heuristics- not a real life case.

The problem emerges as a link capacity design problem for fully restorable ATM networks, which is named as the 'survivable capacity and flow assignment' (SCFA) problem. Joint optimization of capacity and flow assignment is necessary since they are mutually dependent. That is, optimizing one without considering the other would not provide a solution for rapid restoration purpose. Use of two schemes, line and end-to-end restoration, would also reveal the economic gain provided by the end-to-end restoration scheme.

The structure of the SCFA problem necessitates a detailed modeling. An ATM network is modeled as a directed graph $G = (V, A, \tilde{c})$ where V is a node set representing ATM switches, A is a set of directed arcs representing optical trunks, and $\tilde{c} = (c_a)$ is a vector of arc capacity $(a \in A)$.Considering a commodity as a traffic flow from a source to a destination, one commodity is defined for each pair of source and destination nodes.

Let Π be a set of commodities in the network and $Q = (q^{\pi})$ be a vector of requested bandwidth for each commodity $\pi \in \Pi$. Many virtual paths are founded for each commodity to satisfy the traffic requirement. One assumption of the model is that the number of virtual paths per commodity is relatively large compared to the number of commodity restoration flow paths. Considering a widely spread multi-functioning network, this assumption does not form any inconsistency.$\tilde{f}$ denoting a vector of arc flow, SCFA problem can be stated as an optimization problem as follows:

Given	V,A, and Q,	(**SCFA**)
Minimize	Network resource installation cost $D(\tilde{c})$	
over	$\tilde{c}$ and $\tilde{f}$,	
subject to	a) $\tilde{f}$ is a multicommodity flow satisfying Q,	(flow conservation law)
	b) capacity constraints are satisfied, and	(capacity constraints)
	c) full restorability is assured through fast restoration	(full restorability constraints)

Implementing two different schemes on this model, results in two similar linear programming problems.

2 Mathematical Description of the Model

2.1 SCFA-LINE Problem

The first LP model employs line restoration to optimize capacity and flow assignment for ATM networks. With the assumption of network resource installation cost is a linear function of arc capacity, the SCFA-LINE problem is a linear programming problem. Another assumption of the SCFA-LINE's LP problem is that all unit arc installation costs are nonnegative-which is reasonable.

The objective function of the SCFA-LINE problem minimizes the sum of arcs multiplied with their unit installation costs, which is, in fact the network installation cost. First set of constraints (1) state that the demand of a commodity should be equal to the sum of commodities on all virtual paths. The set of capacity constraints (2) guarantee that the spare capacity of each arc must be at least equal to the amount of the arc bandwidth required for restoration from any single link failure. The constraints are expressed for every combination of a link failure event and a remaining arc after the failure. Full restorability constraints (3) provide restoration of all failed arc flows by defining a Boolean type variable $\theta_{a,p}$.

The following notations are used in the formulation of the SCFA-LINE problem:

P^{π}	set of all possible routes for commodity $\pi \in \Pi$;
RP^{a}	set of all possible restoration routes for arc $a \in A$;
$Q_{a,b}$	arc-path indicator variable which equals one if arc a is contained in path b, 0 otherwise;
$\tilde{x}$	$= (x_p^{\pi})\,(\forall \pi \in \Pi, \forall p \in P^{\pi})$. Commodity path flow vector;
$\tilde{r}$	$= (r_p^{a})\,(\forall a \in A, \forall p \in RP^{a})$. Arc Restoration path flow vector;
$\tilde{z}$	$= (z_a)\,(\forall a \in A)$. Spare capacity vector;
E_a	$\equiv \{l \in E : a \notin l\}$. Set of all links except for the one containing arc a;
d_a	$(\forall a \in A)$. Unit arc installation cost, assumed to be nonnegative;
f_a	$(\forall a \in A)$. Amount of a flow of arc a;
z_a^l	$(\forall a \in A, \forall l \in E_a)$. Amount of necessary spare bandwidth of arc a upon a failure of link l;
σ_{π}	$(\forall \pi \in \Pi)$. Price of commodity (dual variable);
w_a	$(\forall a \in A)$. Price of arc restoration (dual variable);
μ_a^l	$(\forall a \in A, \forall l \in E_a)$. Price of arc per failure (dual variable);

The LP problem and its dual are formulated as follows:

Minimize $$D(\tilde{x}, \tilde{r}, \tilde{z}) = \sum_{a \in A} d_a . c_a \qquad \textbf{(SCFA-LINE)}$$

$$
\begin{array}{lll}
\textbf{over} & & \tilde{x} \geq 0, \tilde{r} \geq 0, \tilde{z} \geq 0 \\
\textbf{subject to} & \text{a)} & \sum_{p \in P^\pi} x_p^\pi = q^\pi \qquad \forall \pi \in \Pi \qquad (1) \\
& \text{b)} & z_a - z_z^l \geq 0 \qquad \forall a \in A, \forall l \in E_a \qquad (2) \\
& \text{c)} & \sum_{a \in RP^a} r_p^a - f_a = 0 \qquad \forall \pi \in \Pi \qquad (3) \\
& & c_a = f_a + z_a \\
& & f_a = \sum_{\pi \in \Pi} \sum_{p \in P^\pi} \theta_{a,p}.x_p^\pi \\
& & z_a^l = \sum_{b \in l} \sum_{p \in RP^b} \theta_{a,p}.r_p^b
\end{array}
$$

$$
\begin{array}{lll}
\textbf{Maximize} & & \sum_{\pi \in \Pi} q^\pi.\sigma^\pi \qquad (\textbf{SCFA-LINE-DUAL}) \\
\textbf{over} & & \sigma^\pi, w_a unrestricted, \mu_p^l \geq 0 \\
\textbf{subject to} & \text{i)} & \sum_{a \in p} (d_a + w_a) \geq \sigma^\pi \qquad \forall p \in P^\pi, \forall \pi \in \Pi \qquad (4) \\
& \text{ii)} & \sum_{b \in p:l} \mu_b^l \geq w_a \qquad \forall p \in RP^a, \forall a \in A \qquad (5) \\
& \text{iii)} & \sum_{l \in E_a} \mu_a^l \geq d_a \qquad \forall a \in A \qquad (6) \\
& & \mu_a^{l:(a \in l)} \equiv 0
\end{array}
$$

2.2 SCFA-ETE Problem

SCFA-ETE tries to find an optimal capacity and flow assignment for ATM networks by means of end-to-end restoration. Having similarities with SCFA-LINE, it leaves commodities to chose their rerouting paths between the source and the destination nodes.

Objective function of the model minimizes the sum of unit arc installation cost multiplied with the sum of the spare capacity of each arc and the amount of the flow of that arc. Just like the SCFA-LINE model, flow conservation law (7) enforces the demand of a commodity to be equal to the sum of commodities on all virtual paths. Apart from SCFA-LINE model, it takes into account the amount of bandwidth of an arc for capacity constraints (8).

The additional notation is given below:

- $RP^{\pi,l}$ set all possible relation routes for commodity $\pi \in \Pi$ against failure of link $l \in E$. Its element is called (π, l) *restoration path*;
- $\theta_{a,l,p}$ indicator variable which equals one if both arc a and link l are contained in the path p, and 0 otherwise;

$\tilde{r}$	$= (r_p^{\pi,l})$. Restoration path flow vector $\pi \in \Pi, l \in E, p \in RP^{\pi,l}$;
r_a^l	$\forall a \in A, \forall l \in E_a$. Amount of bandwidth of arc a which is released by affected VP's due to a failure of link l;
$w^{\pi,l}$	$\forall \pi \in \Pi, \forall l \in E$. Price of link restoration per commodity (dual variable).

LP formulation and its dual are as follows:

Minimize $\quad D(\tilde{x}, \tilde{r}, \tilde{z}) = \sum_{a \in A} d_a.(f_a + z_a)$ $\quad$ **(SCFA-ETE)**

over $\quad \tilde{x} \geq 0, \tilde{r} \geq 0, \tilde{z} \geq 0$

subject to

a) $\sum_{p \in P^\pi} x_p^\pi = q^\pi \qquad \forall \pi \in \Pi$ $\quad$ (7)

b) $z_a + r_a^l - z_z^l \geq 0 \qquad \forall a \in A, \forall l \in E_a$ $\quad$ (8)

c) $\sum_{a \in RP^{p,l}} r_p^{\pi,l} - \sum_{a \in l} \sum_{p \in P^\pi} \theta_{d,p}.x_p^\pi = 0$ $\quad$ (9)

$\forall \pi \in \Pi, \forall l \in E$

$$f_a = \sum_{\pi \in \Pi} \sum_{p \in P^\pi} \theta_{a,p}.x_p^\pi$$

$$z_a^l = \sum_{\pi \in \Pi} \sum_{p \in RP^{p,l}} \theta_{a,p}.r_p^{\pi,l}$$

$$r_a^l = \sum_{\pi \in \Pi} \sum_{p \in P^\pi} \theta_{a,l,p}.x_p^\pi$$

Maximize $\quad \sum_{\pi \in \Pi} q^\pi.\sigma^\pi$ $\quad$ **(SCFA-ETE-DUAL)**

over $\quad \sigma^\pi, w_a unrestricted, \mu_p^l \geq 0$

subject to

i) $\sum_{a \in p:(l)} (d_a + w^{\pi,l}) - \sum_{l \in p} \sum_{a \in p} \mu_a^l \geq \sigma^\pi$ $\quad$ (10)

$\forall p \in P^\pi, \forall \pi \in \Pi$

ii) $\sum_{a \in p} \mu_a^l \geq w^{\pi,l}$ $\quad$ (11)

$\forall p \in RP^{\pi,l}, \forall \pi \in \Pi, \forall l \in E$

iii) $\sum_{l \in E_a} \mu_a^l \geq d_a \qquad \forall a \in A$ $\quad$ (12)

$\mu_a^{l:(a \in l)} \equiv 0$

3 Discussion

With clearly defined variables, and effective set of constraints, the models can be considered as well built and good representatives of usage of LP in the field of communication networks. The main drawback of these models is that their complexity requires highly intensive computers for solving them. However, some modifications put the problem in a special form making it easily solvable by revised simplex method. The most important change done for reaching a solvable problem is to use the arc-chain flow representation instead of arc-node representation. This would lower the size of the constraint set especially for large networks.Employing revised simplex method, modified model gives surprisingly good result - much more easier to solve-since the basis matrix appears as an factorizable (LU form) one.

Also assumptions of the models form certain limitations. Both models perform better for large networks since large networks has more virtual paths than the number of restoration flow paths.

Another assumption is considering network installation cost as a linear function of arc capacity. Though a concave cost function is more suitable due to the economies of scale, assuming its linearity does not deteriorate the problem much.

In conclusion, application of linear programming to optimization of ATM networks' capacity and flow assignment brings out the difference between line and end-to-end restoration schemes in terms of cost minimization.

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Linear Programming Model For Finding Optimal Roadway Grades That Minimize Earthwork Cost

by N. Őzturk and İ. Akgün

1 Verbal Description of the Model

In roadway design, there are mainly two stages called the preliminary and final. The preliminary stage is related to the selection of roadway grades, while the final stage is that of minimizing the earthwork involved after preliminary stage is completed.

In the preliminary stage, the position of the grade lines is established. There are several aspects to be considered in grade control so that the selected grades satisfy the geometric specifications of the road. The geometric specifications are related to the elements of the vertical alignment, elevation of grade line at specified stations, horizontal and vertical alignment relations, and type of vertical curve. A common method of laying out the grade line is to insert pins on a sheet of ground profile at the intersection points of the proposed vertical curves, along with a thread drawn taut from the ends of the study section. (A section is the portion of the road between two stations.) By shifting the pins, one can study the effect of different grades without drawing lines on the paper (Wright and Paquette, 1979).

For the selected grade line, the final stage includes establishing the cross section, computing earthwork quantities, and the preparation of earthwork allocations and other construction details. The earthwork can take the form of either excavation in the form of cuts or the construction of embankments in the form of fills. Normally in a road design project, both will be necessary and movement of earth from one part of the site to the other will be required. This should be done with as little waste created or as little extra material required since disposal or collection is expensive.

Among several methods developed for earthwork allocations, the mass diagram is notable. It is a graphical tool for determining the minimum amount of haul and economic distribution of cut and fill. However, the mass diagram has some shortfalls: (1) It is not particularly applicable when hauling cost is not linearly proportional to the hauling distance, (2) It does not incorporate all relevant effects of soil characteristics such as the effect of swell (or shrinkage), and (3) When additional quantities of materials may be borrowed from borrow pits or wasted at landfills (disposal sites), the capacities and costs associated with the elements cannot be considered simultaneously in the mass diagram.

To overcome the previous limitations of mass diagram, several methods employing linear programming (LP)have been developed by Stark and Nicholas (1972), Mayer and Stark (1981), and Nandgaonkar (1981). These models generally compute the earthwork allocations that minimize cost and they assume that roadway grades are fixed. However, the Transport and Road Research Laboratory in the United Kingdom established that the cost of earthwork involved in road construction can vary widely based on the roadway grades chosen by the designer. This has started the studies to integrate the preliminary and final stages in order to reach a solution that will find both optimal grades and earthwork allocations.

One of the studies to integrate two stages is by Easa (1988) who tried to find the complete enumeration of all technically feasible grades, calculating the cut and fill for each grade and then using linear programming to optimize the earthwork involved. However, his approach cannot be considered an integration of two design stages in a true sense. It is actually a trial and error type of approach. Because there are too many feasible grades, global optimality is guaranteed only through an exhaustive consideration of all possible grades.

The work by Moreb (1995), which is actually based on the model developed by Easa, combines the roadway grade selection stage with earthwork allocation stage in a single linear programming problem, thus guaranteeing global optimality.

To represent the problem as a linear program, a road is divided into a number of small sections of equal length. Each section i is considered as a point (x_i, y_i) in the $x-y$ plane. The x-coordinate is the center of this section and the y-coordinate is the average height of the ground in the section. The number of grade lines in the road under study is also determined by the designer. Then, the problem reduces to fitting each grade line (straight line) to some scattered points in the $x-y$ plane using linear programming.

In the model, the objective is to minimize the total cost of excavation, embankment, and hauling. The excavation costs are defined as between-sections and from-borrow-pit costs and the embankment costs as between-sections, from-borrow-pit, and disposal-to-landfill. Hauling cost is the cost of moving earth across sections.

One of the constraints is to ensure that whatever is transported to or from a section is the result of excavation and embankment done to that section. That is, flow-balance constraints are satisfied for each section. The next set of constraints require that the gap between the original height (elevation) of section i and the height of section i after final leveling of that section by grade line j is an exact result of the addition and removal that has been made to section i. To ensure the continuity of the road, constraints that make the consecutive grade lines touch each other are also required. Moreover, to satisfy the geometric specifications of the road, upper and lower limits for the slope and y-intercept of the grade lines are needed.

2 Mathematical Description of the Model

2.1 Indices

n=number of sections in the road
m=number of grade lines (linear segments of the final road)

2.2 Data

2.2.1 General Parameters

A_i =surface area of section i
h_i =the actual average height of section i (as if it is leveled with zero slope)
x_i =the midpoint of section i

2.2.2 Cost Parameters

c_{ij} =cost of transporting one cubic unit of earth from section i to section j
p_i =cost of excavating one cubic unit of earth from section i
q_i =cost of embanking one cubic unit of earth to section i

2.2.3 Grade Line Parameters

L_{1j} =the lower limit of the slope of grade line j
w_{ij} =the upper limit of the slope of grade line j
L_{2j} =the lower limit of the y-intercept value of grade line j
w_{2j} =the upper limit of the y-intercept value of grade line j

2.3 Decision Variables

u_i =the height of earth to be removed from section i
v_i =the height of earth to be added to section i
T_{ij} =number of cubic units of earth to be transported from section i to j
a_j =the slope of grade line j
b_j =the y-intercept of grade line j

2.4 Formulation

$$
\begin{aligned}
\text{Minimize } Z \quad &= \textstyle\sum_{i=1}^{n}(p_i A_i u_i + q_i A_i v_i) \\
&+ \textstyle\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{m} c_{ij}T_{ij}
\end{aligned}
$$

subject to

$$\textstyle\sum_{i=1,i\neq j}^{n}(T_{ij} - T_{ji}) = A_i(u_i - v_i) \quad , i = 1, 2, ..., n$$

$$h_i - (a_j x_i + b_j) = u_i - v_i \quad , i = 1, 2, ..., n, j = 1, 2, ..., m$$

$$a_{j1}x_t + b_{j1} = a_{j2}x_t + b_{j2}$$

$$L_{1j} \leq a_j \leq w_{1j} \quad , j = 1, 2, ..., m$$

$$L_{2j} \leq b_j \leq w_{2j} \quad , j = 1, 2, ..., m$$

The objective function minimizes the total cost of excavation, embankment, and hauling. The first set of constraints ensure that whatever is transported to or from a section is the result of excavation and embankment done to that section. The second set of constraints require that the gap between the original height (elevation) of section i and the height of section i after final leveling of that section by grade line j is an exact result of the addition and removal that has been made to section i. The third set of constraints ensure that the end of every grade line touch the beginning of the following grade line. The above formulation assumes that grade line j_2 meets grade line j_1 at the point x_t. The fourth and fifth set of constraints set upper and lower bounds for the slope and y-intercept of the grade lines, respectively.

3 Discussion

Easa's method uses an iterative approach, where in each iteration, the slopes of the grade lines are incremented or decremented in value. At each iteration a linear transportation problem is solved. Therefore, it only gets closer to the optimum solution, depending on how small his iterative steps are, but can never guarantee global optimality.

The paper that we consider presents a model that succeeds not only in reducing the number of times the problem is solved, but also in arriving at global optimality in solving the problem only twice. The first run is to determine the cut and fill sections, the second run is to find the optimum values of the grade lines and the optimum cost of transporting the cut and fill quantities.

Perhaps the biggest advantage of this model is that it avoids the nonlinearity that is encountered by Easa's approach. A cut or fill in Easa's approach changes in height as well as in width for any change in the grade line, thus resulting in a nonlinear cost component in the objective function. This will lead to the possibility of having local optima, similar to what Easa encountered in his paper. In this paper, however, the section is regarded as a point (x_i, y_i) in the $x - y$ plane representing a fixed width of a section and only the height is allowed to vary in the linear programming model. This implies that as the grade lines change, only the height of a section changes, resulting in a linear cost component in the objective function. The price to pay for this simplification is that a section may turn out to be both a cut and a fill at the same time, in which case the cost of work done inside the section is not included. However, this can be easily taken care of by one of two methods. The first method is by finding the cost incurred within this section separately, and this cost is then added to the total cost. The second method is splitting this section into two new sections, such that one is entirely a cut and the other is entirely a fill. Both methods seem to be effective in practice.

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The Existence of Feasible Schedule in Input Queued Switch Subject to Deadline Constraints: Linear Programming Approach

by I. A. Rai

1 Switching Background

A generic switch consists of input ports, a switch fabric and output ports. The number of input ports and output ports determine the switch size. In general, the number of input ports is equal to the number of output ports and have equal capacities. Figure 1 illustrates the architecture of a generic switch.

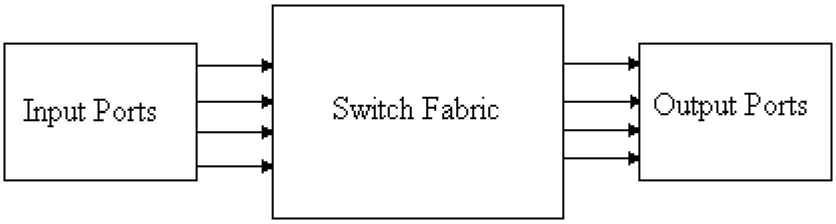

Figure 1: A Generic Switch.

Switch fabric is used to transmit information in the form of fixed size entities called packets from input ports to output ports. A switch fabric is said to be non-blocking if it can transmit packets from different input ports to different output ports simultaneously in a single time unit called *time slot.* If a switch fabric of a generic switch transmits packets going to the same output port in the same time slot a conflict occurs. To avoid this, storage devices are required to store packets at input ports before they are transmitted to output ports or to store them at output ports after transmission. These storage devices are called *buffers* or *queues.* Output buffers are required if the switch fabric can transmit more than one packet to the same output port in the same time slot. This phenomenon of a switch fabric is called *speedup.* If the fabric speedup is S, then it can transmit S packets to the same output port in the same time slot. The output buffered switches have speedup equal to N and the speedup of input buffered switches is 1, where N is the switch size.

Output buffered switches have been traditionally used in networks. Due to the increase in links speed as a result of advancements of fiber optic technology, switch fabrics are required to be very fast. As N increases, practically fast switch fabrics are not available or very expensive. This leads to diversion of research attention to input buffered switches. In these switches packets are first stored at input ports waiting for their scheduled time for transmission. A relatively slower switch fabric is then used to transmit packets to output ports. Since there are no buffers at the output, only one packet can be transferred to one output port in a single time slot.

Head of line (HOL) blocking occurs, when a packet at the front of the queue blocked due to another packet contending the same output port also blocks all the packets behind it in the queue even though those packets may be destined for an output that is currently idle. HOL blocking limits the input buffered switch's throughput to 58.6% when First In First Out (FIFO) service discipline is used. The problem has been recently solved by using *Virtual Output Queueing* (VOQ) technique illustrated in Figure 2.

In VOQ, each input maintains a separate queue for each output. This completely eliminates HOL blocking problem and the throughput is increased to 100%. The problem in input queued switches remains to proposing implementably simple and very fast scheduling algorithms that can efficiently operate in very high speed broadband networks in which Quality of Service (QoS) guarantees are necessary. QoS requirements include guaranteeing delay bounds , bandwidth requested , jitter, fairness, etc.

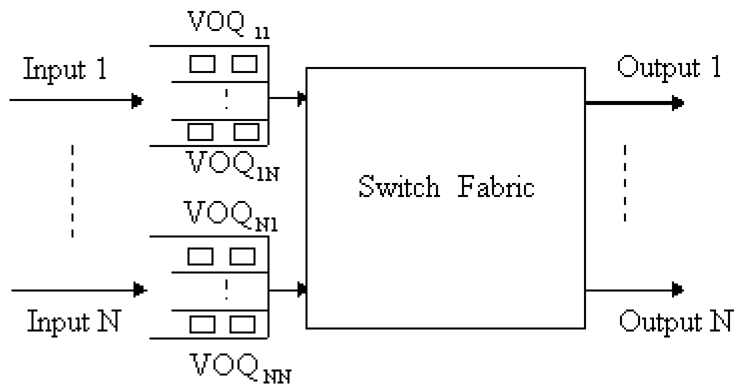

Figure 2: Illustration of VOQ.

In this project, we consider a VOQ input buffered switch subjected to delay constrained traffic streams. In VOQ switch, each input queue has N

sub-queues, where N is the number of output ports. The status of these subqueues can be represented by N×N matrix. We call this matrix a *traffic matrix*.

The objective of this project is to check if this switch can always satisfy the delay constraints of a given traffic matrix. The delay constraints are represented in terms of deadlines of individual packets during transmission. The schedule in which all deadlines represented by a traffic matrix are satisfied is called a *feasible schedule*.

Philp and Liu [1] conjectured that, given periodic traffic with deadlines of packets equal to the period of their traffic streams a feasible schedule always exists if link utilizations are no more than unity. In [2], Giles resolved the conjecture only when periods are evenly divisible, i.e, any period divides all longer ones in the traffic matrix. In our recent work [3], we provide a problem formulation for a more strict version of traffic streams (quasi-periodic) in which the deadlines can be smaller than their streams' periods. Analytical and experimental results in [3] are very close to satisfying the conjecture. Resolving the conjecture analytically is a difficult task, so in this project, we provide the LP feasibility formulation of the problem and search for the traffic matrix which provides a counter example to the conjecture if one exists. In the next section, we provide an LP formulation of the problem.

2 Mathematical Model

Consider an $N \times N$ input queued packet switch. The service requirement on the switch is given by a *traffic matrix*, $M = [m(i,j)]_{N\times N}$, which is a nonnegative integer matrix such that for some integer P

$$\sum_{i=1}^{N} m(i,j) = P \qquad \text{for all } j = 1, 2, \cdots, N$$
$$\sum_{j=1}^{N} m(i,j) = P \qquad \text{for all } i = 1, 2, \cdots, N.$$

Let the *deadline sequence* for a connection (i,j) be the one-sided binary sequence $(d_1(i,j), d_2(i,j), \cdots)$ such that $\sum_{l=1}^{k} d_l(i,j)$ is the smallest number of services that should be received by the connection by the end of time slot k. In particular

$$d_l(i,j) = \begin{cases} 1 & \text{if } l = \lceil kP/m(i,j) \rceil \text{ for some } k \\ 0 & \text{otherwise.} \end{cases}$$

Note that the deadline sequence is periodic, and its period is equal to the schedule length P.

Packet transmission schedule through the switch at any time slot is represented by a *permutation matrix* $\Pi = [\pi(i,j)]_{N\times N}$, which is a binary matrix with exactly one nonzero entry in each row and in each column. Namely, $\Pi(i,j) = 1$ indicates that connection (i,j) receives service in the associated time slot. In that case all other entries on the ith row and jth column are zero, thus each port is served at most once at each time slot. If a sequence $(\Pi_1, \Pi_2, \cdots)$ of permutation matrices is adopted so that the switch is scheduled according to Π_l at each time slot l, then $\sum_{l=1}^{k} \Pi_l(i,j)$ denotes the number of services received by connection (i,j) by time slot k. Thus such a sequence constitutes a feasible schedule if and only if for each connection (i,j),

$$\sum_{l=1}^{k} \Pi_l(i,j) \geq \sum_{l=1}^{k} d_l(i,j) \qquad \text{for each } k = 1, 2, \cdots \tag{1}$$

Note that if such a sequence exists, then the periodicity of the deadline sequence implies that it can be taken periodic by setting $\Pi_k = \Pi_{(k \text{ modulo } P)+1}$. Subtracting both sides of condition (1) from this common value, and defining the matrix $D_l = [d_l(i,j)]_{N\times N}$ yield the following equivalent matrix form of condition (1).

$$\sum_{l=k}^{P} D_l(i,j) \geq \sum_{l=k}^{P} \Pi_l \qquad \text{for each } k = 1, 2, \cdots, P. \tag{2}$$

Here matrix inequalities are understood to hold componentwise. Let's define x_{ij}^{l} as;

$$x_{ij}^{l} = \Pi_l(i,j) \qquad \text{for each } (i,j) \text{ pair and } l = 1, 2 \cdots, P. \tag{3}$$

Similarly, by definition $\sum_{l=1}^{k} D_l(i,j)$ is expressed as,

$$\sum_{l=1}^{k} D_l(i,j) = \lfloor k\frac{m_{i,j}}{P} \rfloor, \qquad \text{for each } k = 1, 2, \cdots, P.$$

But $\sum_{l=1}^{P} D_l(i,j) = m_{i,j}$, which directly implies that,

$$\sum_{l=k}^{P} D_l(i,j) = m_{i,j} - \lfloor (k-1)\frac{m_{i,j}}{P} \rfloor. \tag{4}$$

Let's define the righthand side of Equation 4 to be b_{ij}^k and then substitute Equation 3 and Equation 4 in Equation 2, the following relation follows;

$$x_{ij}^l \leq b_{ij}^k.$$

This is one type of constraints of the problem. Other basic constraints are obtaining permutation matrices Π_l's, that is the row sums and column sums of solution at each l are identical and equal to 1. These constraints can be mathematically represented as follows, for each $l = 1, 2, \cdots, P$;

$$\begin{aligned} \sum_{i=1}^{N} x_{ij}^l &= 1, \qquad \forall j = 1, 2, \ldots, N \\ \sum_{j=1}^{N} x_{ij}^l &= 1 \qquad \forall i = 1, 2, \ldots, N. \end{aligned} \tag{5}$$

For each k and $l = 1, 2, \cdots, P$, the LP formulation of the problem directly follows as shown in 6;

$$\begin{aligned} &\textbf{min } \mathbf{0} \\ &\textbf{subject to} \\ x_{ij}^l &\leq b_{ij}^k \quad \forall (i,j) \text{ pairs} \\ \sum_{i=1}^{N} x_{ij}^l &= 1, \quad \forall j = 1, 2, \ldots, N \\ \sum_{j=1}^{N} x_{ij}^l &= 1, \quad \forall i = 1, 2, \ldots, N \\ 0 &\leq x_{ij}^l \leq 1 \quad \forall (i,j) \text{ pairs}. \end{aligned} \tag{6}$$

As it can easily be seen, the problem is a feasibility problem in which no *objective function* is required to be optimized. If a feasibility problem results to optimal solution, then the constraints of the problem are satisfied. In case of our problem, we try to guarantee deadline constraints of packets in a schedule. For each $l, j = 1, 2, \cdots, PN^2$ and $k, iand = 1, 2, \cdots, PN^2 + 2PN$, let $\hat{x}_{ij}^l$ be a column vector of all PN^2 variables and let $\hat{b}_{ij}^k$ be a column vector defined as follows;

$$\hat{b}_{ij}^k = \begin{cases} m_{i,j} - \lfloor (k-1)\frac{m_{i,j}}{P} \rfloor & \text{if } k = 1, 2, \cdots, PN^2 \\ 1 & \text{otherwise.} \end{cases}$$

The above LP formulation can then be expressed in compact form as shown in 7;

$$
\begin{array}{c}
\textbf{min 0} \\
\textbf{subject to}
\end{array}
$$

$$
\begin{aligned}
A\hat{x}^l_{ij} &\leq \hat{b}^k_{ij}, \\
0 &\leq \hat{x}^l_{ij} \leq 1 \quad \text{for some } (i,j) \text{ pairs.}
\end{aligned} \tag{7}
$$

In general this LP formulation provides integer solution if the matrix A is *unimodular* [4]. It is not an easy task to determine whether the matrix A is unimodular or not analytically. However, as will be discussed in Section 3, simulations reveal that the formulation does not guarantee integer solution and so the matrix A can be concluded as not being unimodular. Since we need integer solution we present an *Integer Linear Programming* (ILP) for each k and $l = 1, 2, \cdots, P$ as follows.

$$
\begin{array}{c}
\textbf{min 0} \\
\textbf{subject to}
\end{array}
$$

$$
\begin{aligned}
x^l_{ij} &\leq b^k_{ij}, \quad \forall (i,j) \text{ pairs} \\
\sum_{i=1}^{N} x^l_{ij} &= 1, \quad \forall j = 1, 2, \ldots, N \\
\sum_{j=1}^{N} x^l_{ij} &= 1, \quad \forall i = 1, 2, \ldots, N \\
x^l_{ij} &= \{0, 1\} \quad \forall (i,j) \text{ pairs.}
\end{aligned}
$$

The formulated LP problem takes a traffic matrix as an input which results to $PN^2 + 2PN$ constraints and PN^2 variables denoted as x^l_{ij}. The values of b^k_{ij} can be directly obtained from traffic matrix. AMPL and CPLEX mathematical programming languages are used to simulate the above LP model as formulated in this section. The discussion of simulations is presented in the next section.

3 Simulations and Discussion

The formulated problem was implemented in AMPL and CPLEX mathematical programming packages. Pure LP formulation of the model provides some real values when *minos* or *lpsolver* solvers are used but we require the solution to be integer (binary). Minos provides real solution to binary formulation as well, this is due to the fact that minos does not solve integer programming problems. The lpsolver is used instead of minos to solve the binary formulation and results to binary solution of the output. The AMPL program can be seen in Appendix A. Available AMPL and its associated solvers however, are limited to small number of variables and constraints. For example, AMPL fails for values of $N = 4$, $P = 16$ and above. This intern limits the scope of our experiment. Therefore, we implement the binary formulation of the problem in CPLEX mathematical programming. The CPLEX program can be seen in Appendix B.

Using CPLEX we are able to test larger dimensional traffic matrices. Up to the time of writing this report however, we have to come across any traffic matrix which does not provide optimal solution which further enlightens us the existence of feasible schedule of any given traffic matrix and that Philp and Liu conjeture may indeed be true.

Consider the following example of traffic matrix representing switch dimension of 2 and schedule length $P = 3$.

Example 1

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The feasible solution for AMPL with minos for both; pure LP and binary LP is;

$$\Pi_1 = \begin{bmatrix} 1 & 2.43238e-16 \\ 2.43238e-16 & 1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

As can be seen in this simple example, some values are real although negligible. But it was noted during simulations for $N = 4$ and $P = 8$ that some solutions are large real values. The feasible solution for AMPL with lpsolver and CPLEX are binary for binary formulation as seen below ;

$$\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In general the model is useful and may give us more intuition on the problem. The LP model is certain and gives the solution whenever it exists. While searching for counter examples, the programs also provide the schedules too, i.e, a set of permutation matrices. Studying the pattern of these permutation matrices and its relation to input traffic matrix may even give rise to proposing better heuristic algorithms.

4 Summary

In this project, we are using linear programming techniques to determine if Philp and Liu conjecture is true. We present a feasibility binary LP formulation and implement it in mathematical programming languages such as AMPL and CPLEX. Since AMPL version we have is limited to a small number of variables and constraints we put more emphasis on CPLEX package in our simulations.

We run the CPLEX program to all traffic matrices which failed to provide feasible schedules to the proposed algorithms in [3]. The main objective is to see if there is a traffic matrix which does not provide a feasible schedule, i.e, which does not results to optimal solution of our LP formulation hence a it is a counter example. Up to the the time of writing this report, we have not yet encounter any counter example.

Finally, LP formulation seems to be a very useful tool to resolving infeasibility problems since it always satisfies the feasibility constraints if they are so under given input and constraints and provides infeasible solution if the input to formulation is infeasible under given inputs and constraints.

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A Fast Linear Programming for Blind Equalization

by Ő. Selvi and S. Çevik

1 Verbal Description of the Model

Blind equalization of an unknown, non-minimum phase channel is an important problem in data communication systems. A blind equalizer achieves parameter adaptation based on the observed output and priori knowledge of some distributional or statistical properties of the input sequence. By eliminating the need for training signals, a blind equalizer can begin its self- adaptation without disturbing the normal flow of data transmission. For a generic class of channels with QAM(Quadrature Amplitude Modulated) input signals, obtaining the optimal equalizer parameters for achieving ISI(Inter Symbol Interference)suppression is the main goal of the equalizer. ISI is the source of bit errors in a baseband-pulse transmission system, which arises when the communication channel is dispersive. QAM input signals are the form of 2-tuples of bits (00,01,10,11) while input signals are of the form individual bits(zeros and ones). These 2-tuples of input signals are transformed into specific set of complex numbers defined as symbols.This set is $\{-1-j,-1+j,1-j,1+j\}$.The transformation is described as;

$$00 \mapsto (-1-j), 01 \mapsto (-1+j), 10 \mapsto (1-j), 11 \mapsto (1+j)$$

.

In a baseband representation of the QAM channel equalization system, the original symbol sequence $\{a_k\}$ takes values from a complex signal set $\mathcal{A}$ (known as the QAM constellation) in a manner such that all possible finite symbol sequences occur with non-zero probability. The complex channel input $\{a_k\}$ is transmitted through a non ideal channel, assumed to be linear, casual and (bounded-input-bounded-output) stable. The channel transfer function can be written as

$$H(z^{-1}) = \sum_{i=0}^{\infty} h_i z^{-i}, \quad h_i \in C$$

where C signifies the set of complex numbers and $\{h_i\}$ represents the channel impulse response sequence. When the channel is such that there are more than one nonzero element in $\{h_i\}$ the channel output

$$x_k = H(z^{-1})a_k = \sum_{i=0}^{\infty} h_i a_{k-i}$$

suffers from ISI and the removal of this linear distortion is the task of channel equalizers.

A *linear* channel equalizer is a linear filter $\Theta(z^{-1})$ that is applied to the channel output x_k in order to eliminate the ISI. Initially we may take this filter as being stable and potentially non-casual(doubly infinite), and of the form:

$$\Theta(z^{-1}) = \sum_{i=-\infty}^{\infty} \theta_i(k) z^{-i}, \quad \theta_i \in C$$

so as to deal with non-minimum phase channels. The time dependence of the equalizer parameters signifies that they are subject to adaptation. The equalizer output can then be written as;

$$z_k = \sum_{i=-\infty}^{\infty} \theta_i(k) x_{k-i} = X_k' \theta(k)$$

by denoting the adaptive equalizer parameter vector as

$$\theta(k) = [...\theta_{-2}\ \theta_{-1}\ \theta_0\ \theta_1\ \theta_2...]'$$

and the equalizer input signal regressor as

$$X_k = [...x_{k+2}\ x_{k+1}\ x_k\ x_{k-1}\ x_{k-2}...]'$$

the ideal objective of any channel equalizer $\Theta(z^{-1})$ is to achieve distortionless reception

$$z_k = \Theta(z^{-1})x_k = a_{k-\nu}, \quad \nu \in Z_+,\ \forall k$$

or simply

$$H(z^{-1})\Theta(z^{-1}) = z^{-\nu}$$

where we can tolerate a finite fixed delay ν.(The strict form of this objective will be modified subsequently). Such an objective translates into the identification of channel inverse by the equalizer, *i.e.*,

$$\Theta(z^{-1}) = z^{-\nu}\check{h}(z^{-1})$$

where $\check{h}(z^{-1}) = H^{-1}(z^{-1})$ is understood to be the stable, (potentially) doubly infinite (non-casual) inverse of the channel $H(z^{-1})$. The goal of equalization could be achieved by adapting the equalizer parameters to minimize the MSE(Mean Square Error) between the channel output $a_{k-\nu}$ and the equalizer output, when the original channel input signal is available to the receiver as a training sequence.

In blind equalization, the original sequence is unknown to the receiver except for its probabilistic or statistical properties over the known alphabet $\mathcal{A}$. Usually this signal constellation $\mathcal{A} \subset C$ has symmetrical properties such that

$$\exp(jm\pi/2)\mathcal{A} = \mathcal{A}, \quad m \in \{0, 1, 2, 3\}$$

i.e., the constellation is $m\pi/2$ rotation-invariant. If the statistics of the input data reflects this same symmetry over $\mathcal{A}$, then a $m\pi/2$ phase rotation does not cause any statistical changes in the channel output. Thus, the possible equalization result would be

$$H(z^{-1})\Theta(z^{-1}) = \exp(m\pi/2)z^{-\nu}, \ \ m \in \{0, 1, 2, 3\}$$

which relaxes the objective $H(z^{-1})\Theta(z^{-1}) = z^{-\nu}$. A further relaxation of the ideal objective is necessary since in practical applications only equalizers with a finite number of adjustable parameters can be implemented rather than those of

$$\Theta(z^{-1}) = \sum_{i=-\infty}^{\infty} \theta_i(k) z^{-i}, \quad \theta_i \in C$$

Typically these finite parameters are arranged in the form;

$$\Theta(z^{-1}) = z^{-N} \sum_{i=-N}^{N} \theta_i(k) z^{-i}$$

In designing our new equalizer based on linear programming, we do not attempt to identify the exact gain of channel inverse and focus instead on the elimination of ISI which is the primary objective of channel equalization.

2 Mathematical Description of the Model

We assume QAM constellation is such that the real and imaginary components are independent and identical with

$$M = \max|Re\{a_k\}| = \max|Im\{a_k\}|$$

We can denote the total system (channel and equalizer) as

$$\mathcal{T}(q^{-1}) = H(q^{-1})\Theta(q^{-1}) = \sum_{i=-\infty}^{\infty} t_i q^{-i}$$

where $\{t_i\}$ is the impulse response of the combined system given by

$$t_i = h_i * \theta_i = \sum_{k=0}^{\infty} h_k \theta_{i-k} \quad \text{(* defines a convolution sum)}$$

with equalizer output $z_k = \sum_{i=-\infty}^{\infty} t_i a_{k-i}$. The equalizer is taken to be doubly infinite and non-causal to permit complete removal of ISI. This assumption will be relaxed so that the equalizer remains in the truly implementable finite dimensional parameter space.

The cost function to be minimized is chosen as

$$\begin{aligned} J(\theta) &= \max|Re\{z_k\}| + \max|Im\{z_k\}| \\ &= \max_{\{a_k\}} |\textstyle\sum_i Re\{t_i\}Re\{a_{k-i}\} - Im\{t_i\}Im\{a_{k-i}\}| \\ &+ \max_{\{a_k\}} |\textstyle\sum_i Re\{t_i\}Im\{a_{k-i}\} + Im\{t_i\}Re\{a_{k-i}\}| \\ &= 2M \textstyle\sum_{-\infty}^{\infty} [|Re\{t_i\}| + |Im\{t_i\}|] \end{aligned}$$

The convexity of $J(\theta)$ with respect to θ can be seen as follows: both $Re\{t_i\}$ and $Im\{t_i\}$are linear in θ, therefore $|Re\{t_i\}|$ and $|Im\{t_i\}|$ are convex functions of θ; since $J(\theta)$ is the sum of these convex functions, it is also convex. Without any constraints on θ, the convex cost function achieves a trivial global minimum at $\theta = 0$, with equalizer output $z_k = 0$.

In order to make the cost function $J(\theta)$ useful, a linear constraint

$$Re\{\theta_0\} + Im\{\theta_0\} = 1$$

is proposed on the equalizer parameter vector to avoid the all zero solution without compromising the ability of the equalizer to remove ISI. Due to the linearity of the constraint, the convexity of the cost function is maintained and global convergence is therefore assured.

Linear Programming Formulations

We restrict θ in the form of $\theta = (\theta_{-N}, \theta_{-N+1}, ..., \theta_0, ..., \theta_N)$. Then the dimension of equalizer parameter vector θ is $2N + 1$. Given the block of equalizer input data $\{X_k\}_{k=-K}^{K}$, then the output of the equalizer is $z_k = X_k^{'}\theta$ where $X_k^{'}$ is the signal regressor. Then θ can be found by

$$
\begin{array}{ll}
\text{minimize} & \max_{-K\le k\le K} |Re\{z_k\}| + \max_{-K\le k\le K} |Im\{z_k\}| \\
\text{subject to} & Re\{\theta_0\} + Im\{\theta_0\} = 1
\end{array}
$$

Note that

$$
Re\{z_k\} = Re\{X_k^{'}\theta\} = \sum_{i=-N}^{N} Re\{x_{k-i}\}Re\{\theta_i\} - Im\{x_{k-i}\}Im\{\theta_i\}
$$

and

$$
Im\{z_k\} = Re\{X_k^{'}\theta\} = \sum_{i=-N}^{N} Re\{x_{k-i}\}Im\{\theta_i\} + Im\{x_{k-i}\}Re\{\theta_i\}
$$

Introducing two new variables τ_1 and τ_2, we can reformulate the above minimization problem as the following linear program

$$
\begin{array}{ll}
\text{minimize} & \tau_1 + \tau_2 \\
\text{subject to} & Re\{\theta_0\} + Im\{\theta_0\} = 1 \\
& -\tau_1 \le \sum_{i=-N}^{N} Re\{x_{k-i}\}Re\{\theta_i\} - Im\{x_{k-i}\}Im\{\theta_i\} \le \tau_1 \\
& -\tau_2 \le \sum_{i=-N}^{N} Re\{x_{k-i}\}Im\{\theta_i\} + Im\{x_{k-i}\}Re\{\theta_i\} \le \tau_2, \quad k = 0, \pm 1, \pm 2, .., \pm K
\end{array}
$$

To equivalance of the previous formulation to this formulation follows from the fact that when at optimality, the variables τ_1 and τ_2 must be equal to $\max_{-K\le k\le K} |Re\{z_k\}|$ and $\max_{-K\le k\le K} |Im\{z_k\}|$ respectively.

The above linear program has $4N + 4$ variables and $8K + 5$ constraints. In this formulation our extra variables except τ_1 and τ_2 are $Re(\theta_i)$'s and $Im(\theta_i)$'s.

An alternative formulation of the equalization problem is the following

$$
\begin{array}{ll}
\text{minimize} & \max_{-K\le k\le K} |Re\{z_k\}| \\
\text{subject to} & Re\{\theta_0\} + Im\{\theta_0\} = 1
\end{array}
$$

we can minimize the convex cost

$$
\max_{\{a_k\}} |Re\{z_k\}|
$$

instead of the convex cost

$$\max_{\{a_k\}} |Re\{z_k\}| + \max_{\{a_k\}} |Im\{z_k\}|$$

The equivalent linear programming formulation is the following

$$\begin{array}{ll} \text{minimize} & \tau_1 \\ \text{subject to} & Re\{\theta_0\} + Im\{\theta_0\} = 1 \\ & -\tau_1 \leq \sum_{i=-N}^{N} Re\{x_{k-i}\}Re\{\theta_i\} - Im\{x_{k-i}\}Im\{\theta_i\} \leq \tau_1, \quad k = 0, \pm 1, \pm 2, .., \pm K \end{array}$$

Notice that this linear program has only $4N + 3$ variables and $4K + 3$ constraints. Thus it is simpler than previous formulation and generally leads to faster convergence time for the equalizer. An entirely similar formulation is to minimize the imaginary part of the combined channel output only, leading to the following linear program

$$\begin{array}{ll} \text{minimize} & \tau_2 \\ \text{subject to} & Re\{\theta_0\} + Im\{\theta_0\} = 1 \\ & -\tau_2 \leq \sum_{i=-N}^{N} Re\{x_{k-i}\}Im\{\theta_i\} + Im\{x_{k-i}\}Re\{\theta_i\} \leq \tau_2, \quad k = 0, \pm 1, \pm 2, .., \pm K \end{array}$$

3 Discussion

The baud-rate equalization algorithm presented here is based on a convex cost function coupled with a simple linear constraint on the equalizer parameters. For a generic class of channels with QAM input signals, this algorithm allows the convergence of equalizer parameters to a unique global minimum achieving ISI suppression.

Although other baud-rate equalizers(i.e. Constant Modulus Algorithm, Shalvi-Weinstein Algorithm) are not globally convergent for arbitrary initialization, linearly constrained blind equalizer which is presented here is globally convergent with a convex cost. This algorithm does not attempt to identfy the exact gain of the zero-forcing equalizer and focus instead on generating an equalizer output that is the scaled version of the channel input.

Also, since the minimization of the l_∞ cost are realized via l_p approximation or via iterative block data estimate, the other algorithms are slow and tedious, making them undesirable for practical implementation. However, the algorithm represented here, which is based on a linear programming formulation for minimizing the l_∞ convex cost function, is computationally efficient. In addition, this linear programming approach requires substantially fewer data samples for training and much shorter equalizer length, and it demonstrates a vast improvement of convergence speed.

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Linear Programming for Flood Control in the Iowa and Des Moines Rivers

by B. Uçar

1 Verbal Description of the Model

Reservoir systems are built to control floods, among some other objectives like hydroelectric power production and fish and wildlife maintenance. Due to environmental concerns, people concentrated on optimally operating the currently existing reservoirs rather than designing new projects. The paper by Needham et. al. describes an application of deterministic optimization to assess flood control operations at the Iowa/Des Moines River Reservoir System.

River reservoir systems consist of reservoirs and control points. Reservoirs can be used to store and release water. They are used to regulate the difference between the total in-flow and the desired total out-flow with proper storage or release of water. Control points are locations where we want to control the amount of flow. The flow at control points is referred to as discharge, which is equal to both the in-flow and the out-flow there. From this perspective control points can be seen as storageless reservoirs whose in-flow should be controlled in order to keep the flow at certain levels. Once a flow enters the system it is called a *controlled* flow. At the very first point that an in-flow enters the system it is called an *uncontrolled* flow.

The damage of the floods is proper to the flows at the control points. When the flow exceeds the proper amount a penalty is applied to represent the damage of the flood. The penalties regarding the damage of the floods as considered by the authors, represents the total penalty at each control point which is a combination of urban, rural, and agricultural damages. The flows can be regulated by operating the reservoirs appropriately. The water level at reservoirs can not be modified arbitrarily. First, rapidly changing the release rates causes bank sloughings. Second, response times to changing downstream conditions may be inadequate. Therefore, release rates outside the proper margins bring extra penalty. Another point is that, there is usually a target range for the reservoirs' water levels. Levels that are outside the target range incur extra cost.

Assume we are given the in-flow and out-flow of the reservoirs as well as the control points at a certain time point. Then, the release or storage

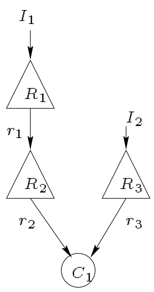

Figure 1: A simple river reservoir system. R_i's represent reservoirs, r_i's represent releases, C_1 represents control point.

at a certain reservoir added to the out-flow will be equal to the in-flow of the reservoir. This is also true for the control points where we have zero storage. Unfortunately, when we consider water flows things become more complicated. Consider a river reach. The in-flow of the reach is equal to the out-flow when we are in the discrete domain, or at the limits in continuous domain. Now assume that we discretized the continuous domain and measured the in-flows and out-flows at time points 1, 2, and 3 as $I(1), I(2), I(3)$ and $O(1), O(2), O(3)$, respectively. Are $\sum_{t=1}^{3} I(t)$ and $\sum_{t=1}^{3} O(t)$ equal? The answer is negative. Fortunately, people used a scheme, so called Muskingum Routing, to relate those values. In the Muskingum Routing scheme, out-flow t is computed as a linear combination of in-flows at times 1 through t. Consider a peak at in-flow at time t. It will contribute to the out-flow at later time points with an attenuation. This attenuation is related to the travel time of the flow. The coefficients used in computing out-flows depend on this attenuation [3].

In a river reservoir system the flows are routed according to Muskingum scheme from outlets of a reservoir or a control point to another reservoir or control point. The reservoirs and control points that route their flow to another one are said to be in *upstream* of the target reservoir. Consider the simple river reservoir system shown in Figure 1 taken from [2]. The in-flow to the control point C_1 at time t is computed as $\sum_{i=1}^{t}(c_{2,i} \times r_{2,i}) + \sum_{i=1}^{t}(c_{3,i} \times r_{3,i})$, where $c_{j,i}$'s are Muskingum coefficients and $r_{i,j}$'s are releases. Accordingly, the in-flow to the reservoir R_2 at time t can be computed as $\sum_{i=1}^{t}(c_{1,i} \times r_{1,i})$. Note that there is only one system element in the upstream of R_2. This proper routing of the flow through the system is referred to as continuity constraint.

Optimizing the operations of river-reservoir systems for flood control is meant to be minimizing the total system penalties for too much or too little release, storage, and flow summed over a period of time constrained to the continuity of the flow through control points and reservoirs. Therefore, the objective is to find a system-wide set of releases for each reservoir and control point per each time period of interest that minimizes the cumulative penalties.

2 Mathematical Description of the Model

The river reservoir system of the paper can be seen in Figure 2. In the system, we have 3 reservoirs –represented as triangular objects– and 10 control points –represented with circles. Other than the in-flows through the system connections, there is an uncontrolled in-flow associated with each reservoir and each control point, represented with I's.

The LP model proposed by authors, uses given releases to compute the storage and downstream flows and accommodates continuity constraints with Muskingum routing, and accounts for reservoir outlet capacities.

The continuity constraint through the reservoir j at time period i takes the form:

$$\frac{1}{\Delta t}\left[S_{i,j} - S_{i-1,j}\right] + f_{i,j} - \sum_{k \in \Omega_j} \sum_{t=1}^{i} c_{t,k} f_{t,k} = I_{i,j} \tag{1}$$

where $S_{i-1,j}$ and $S_{i,j}$ represents the storage at the beginning and at the end of the period i, respectively. Total release at the period i is represented with $f_{i,j}$. Ω_j represents the set of reservoirs and control points that route their flows to reservoir R_j. The flow at time period t from a reservoir R_k in set Ω_j is –denoted $f_{t,k}$– routed according to the Muskingum scheme with proper coefficient $c_{t,k}$. The uncontrolled in-flow to the reservoir R_j at time period i is represented as $I_{i,j}$. Note that, the storages at time i at the reservoirs depend on the previous operations. Therefore, the storage levels of the reservoirs at time period 1 are assumed to be known.

Total storage capacities of reservoirs are divided into zones to model the storage schemes. The total storage of a reservoir at time i hence becomes equal to the sum of the storages at each zone, i.e.

$$S_{i,j} = \sum_{l=1}^{NLF} S_{i,j,l} \tag{2}$$

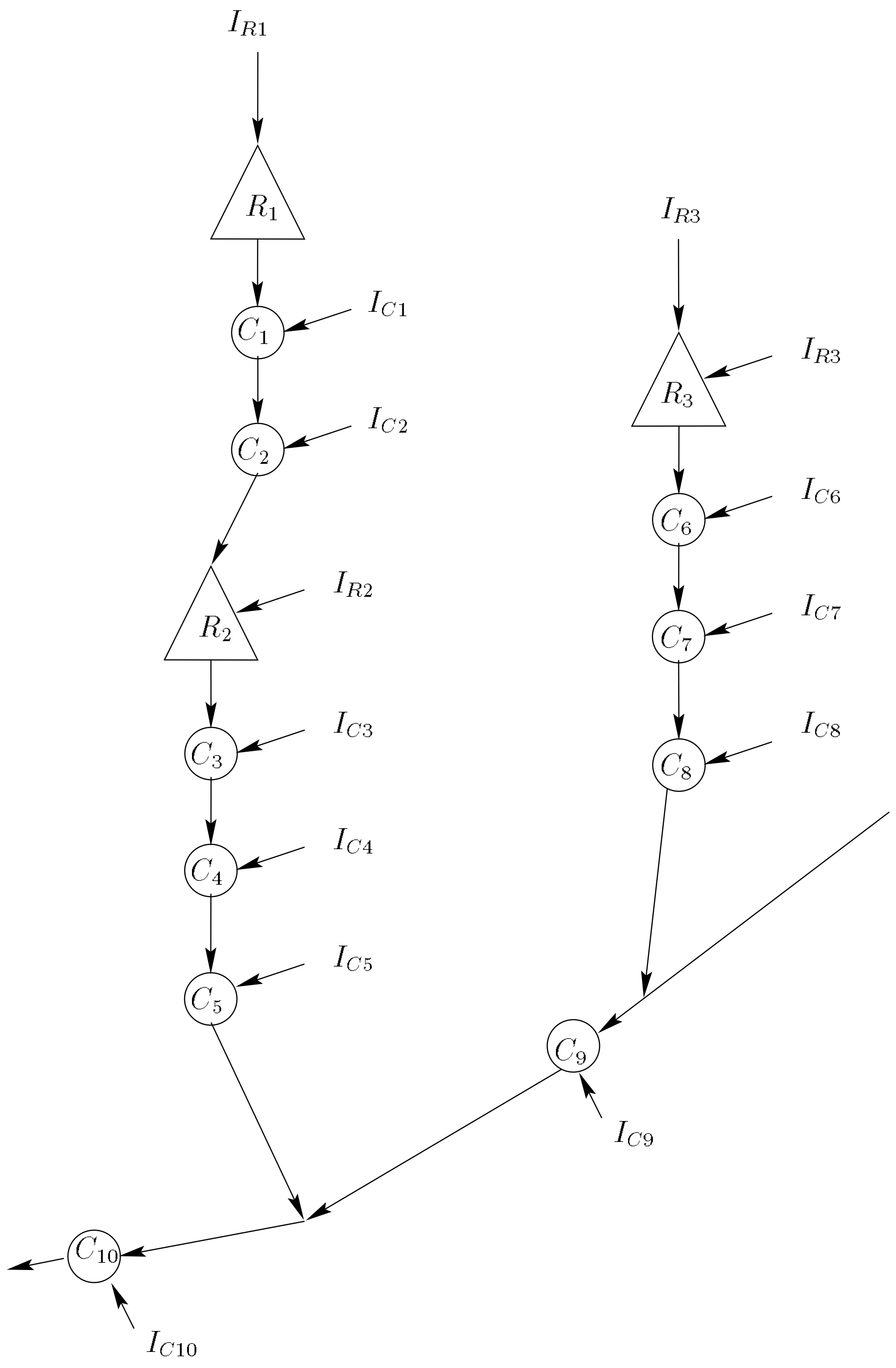

Figure 2: Reservoir, control point system; R_i: reservoir i, C_i: control point i, I: uncontrolled inflows.

where NLF is the number of storage zones. With this setting the continuity constraint above is replaced by

$$\frac{1}{\Delta t}\left[\sum_{l=1}^{NLF} S_{i,j,l} - \sum_{l=1}^{NLF} S_{i-1,j,l}\right] + f_{i,j} - \sum_{k\in\Omega_j}\sum_{t=1}^{i} c_{t,k} f_{t,k} = I_{i,j} \tag{3}$$

The reservoirs in the Iowa and Des Moines Rivers are divided into 5 zones: drought, conservation, flood control, emergency flood control, and surcharge zones. The storage at zone l is constrained as

$$S_{i,j,l} \le S_{j,l}^{max} \tag{4}$$

where, $S_{j,l}^{max}$ represents the maximum allowable storage for zone l of reservoir R_j.

The releases from the reservoirs are physically limited with the characteristics of outlets. The maximum release from a reservoir R_j at time period i is represented as a linear function of the outlets of the zones, i.e.,

$$f_{i,j} \le \sum_{l=1}^{NLF} \frac{\beta_{j,l}}{2}(S_{i-1,j,l} + S_{i,j,l}) \tag{5}$$

where $B_{j,l}$ is the slope of the storage-discharge capacity relation at zone l. Storage-discharge relation become more complicated in forced spill conditions. When storages at certain zones reach their maximum the water spill into the next zone. To catch this scenario, binary variables are introduced for each reservoir j. For example the constraints

$$\begin{array}{rcl} \sum_{l=1}^{2} S_{i,j,l} & \ge & Y_{i,j}\sum_{l=1}^{2} S_{j,l}^{max} \\ S_{i,j,l} & \le & Y_{i,j} S_{j,3}^{max} \\ Y_{i,j} & \in & \{0,1\} \end{array} \tag{6}$$

are used to ensure that water is not stored in zone 3 before zones 1 and 2 are filled. If the water levels at the first two zones are below the limits, the value of $Y_{i,j}$ will be computed as 0, and hence a storage level of zero will be found for zone 3 for feasibility.

The continuity constraint for each control point takes the following form:

$$f_{i,j} - \sum_{k\in\Omega_j}\sum_{t=1}^{i} c_{t,k} f_{t,k} = I_{i,j} \tag{7}$$

where $f_{i,j}$ represents the average flow at time period i for control point C_j. Please note that, a control point is nothing than a storageless reservoir. With this analogy the control points are also divided into zones. In this case, the division is into damage zones. Therefore, the continuity constraint for the control points above is replaced by,

$$\sum_{l=1}^{NF} f_{i,j,l} - \sum_{k \in \Omega_j} \sum_{t=1}^{i} c_{t,k} f_{t,k} = I_{i,j} \tag{8}$$

where NF is the number of zones.

The first part of the cost to be minimized is the penalties for too much or too little storage at reservoirs. These penalties represents the inconvenience of the storage levels outside the target range set by operators. For each reservoir, the penalties are specified as linear function of stored water at each zone during the period. The total storage penalty SP_j for reservoir R_j overall time periods including the i'th one is defined by the equation

$$SP_j = \sum_{t=1}^{i} \sum_{l=1}^{NLF} A_{j,l} S_{i,j,l} \tag{9}$$

where $A_{j,l}$ represents the slope of the penalty function at zone l of the reservoir R_j.

As said previously, releases from the reservoirs can not be arbitrarily set. Rapid changes in release rates damages the vicinity of the reservoirs, or it may cause inadequate response times for the following time periods. To include the penalties related to the release rates, the proposed LP model includes some auxiliary constraints that represent release at time i as a combination of release at time period $i-1$ and plus or minus a change in release. If the absolute value of this change exceeds a specified maximum a penalty is imposed. The auxiliary constraints that relate the release for each period to the previous period's release takes the form

$$\begin{aligned} f_{i,j} = f_{i-1,j} + [Ra^+_{i,j} &+ Re^+_{i,j}] - [Ra^-_{i,j} + Re^-_{i,j}] \\ Ra^+_{i,j} &\leq L^+_{i,j} \\ Ra^-_{i,j} &\leq L^-_{i,j} \end{aligned} \tag{10}$$

where $Ra^+_{i,j}$ and $Re^+_{i,j}$ represent the acceptable and excessive release increase, respectively. Accordingly, $Ra^-_{i,j}$ and $Re^-_{i,j}$ represents acceptable and excessive release decrease, respectively. Acceptable release increase and decrease amounts are constrained to not to exceed user specified limits, $L^+_{i,j}$ and $L^-_{i,j}$. The excessive release increase and decrease values are used to impose a total penalty for reservoir j overall times including the i'th period as

$$RP_j = \sum_{t=1}^{i} B_{i,j} Re^+_{i,j} + \sum_{t=1}^{i} D_{i,j} Re^-_{i,j} \tag{11}$$

where $B_{i,j}$ ($D_{i,j}$) represents penalty per unit flow for a positive (negative) change in release greater than the user specified limits.

The last term in the objective function is the flood damage at control points. Since the flood is related to the flow amount which is the sum of local uncontrolled flow and routed reservoir releases, total penalty is specified as

$$QP_k = \sum_{t=1}^{i} \sum_{l=1}^{NF} E_{k,l} f_{i,k,l} \tag{12}$$

using Equation (8). $E_{k,l}$ is the slope of the penalty function at zone l of the control point C_k.

Using the penalty terms specified with equations in (9), (11), and (12), the objective function is specified as

$$\left[\sum_{k \in \Psi} QP_k + \sum_{j \in \Phi} RP_j + \sum_{j \in \Phi} SP_j \right] \tag{13}$$

where, Ψ and Φ represent the set of all control points and reservoirs, respectively. The decision variables related to the reservoirs are releases from each zone, storage levels at each zone, excessive release increase/decrease from each zone for each time period. The decision variables regarding the control points are flows to each damage zone for each time period. Then the problem is meant to be finding the release schedule that yields the minimum of the total penalty function given by (13) subject to the constraints stated by (10), (8), (6), (5), (4), (3), and (2).

3 Discussion

The usefulness and limitations come together in this river-reservoir system. A limitation is that, the model will be run to decide on the future action plans with forecasted data, such as uncontrolled flow into the system. The forecasted data should be realistic, accurate enough to make decisions. If this is not the case then the model will be useless. Another point is that, the linear routing mechanism and linear penalty functions should be adequate enough to model the system properly. If this is not the case then the model provided will be useless. Although the authors count the forecasted input data and linear functions in the limitations of the model, I think that in practice there are not any problems associated with those. My rationale is that, we have to use forecasted data because we are concerning events in the future. From the results of the application I see that in most of the simulation periods the authors realized little deviations from the storage levels and releases recorded for past events. This coincidence with recorded actions validates the accuracy of the linear routing scheme and linear penalty functions.

To enhance the usability of the model the authors should include sensitivity analysis. They use forecasted data which is not perfect. Therefore, they should provide means to decide not to use the previously found operation rules according to the current data. This may be provided by analyzing the sensitivity of the results to the uncontrolled input flows. This will results in rules like "if the current period's flow is greater than a limit rerun the model to change the action plans". These rules will increase the load of the operators. This increase in the load of operators is detriment to the usability of the model in real life, which I could not find a proper solution. The last thing is also related to the real life. As the authors also stated, it will be hard to convince the public that they should be flooded today to reduce the cost of damages of possible floods in the future.

Nonetheless, I think that the model is useful, at least theoretically.

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A Linear Programming Model For Real-Time Pricing of Electric Power Service

by H. Ümit and P. Tan

1 Verbal Description of the Model

Real-time Pricing (RTP) is an electric power service offering in which prices vary over time, based on projected supply and demand conditions.Prices may vary hourly or in larger time blocks and are typically announced several hours to one day in advance. RTP is an attractive option because it approximates the economic efficiency of spot pricing, while allowing customers time to react to prices.

Advances in metering and control technologies have made it possible to offer differentiated service to a greater base of customers. By smoothing the load and reducing the need for reserve capacity, these service offerings allow generation capacity expansion to be postponed or avoided in some cases. Instantaneous market clearing spot prices are not feasible for electric power because they do not provide the utility or customers sufficient time to react. Supply planning requires the utility to make short- and medium- term adjustments, such as unit commitments, while major supply changes often require negotiated power purchases or long-term capital investments. Similarly, customers must often make short-term adjustments in order to curtail their demand in response to higher prices, and may also require capital investments and other long term modifications to realize their full potential to respond to price changes. Thus, RTP which typically announces prices a day ahead coupled with a contract that specifies limits on yearly price fluctuations, is an attractive option for many electric power applications.

Electric utilities' costs and ability to respond to changing supply requirements have been analyzed extensively. Customers' short- and long-term costs and ability to curtail demand are much less understood and are not included in a comprehensive manner in setting real-time prices. Thus, despite the sound economic foundations for RTP, comprehensive methods for its planning and implementation are currently incomplete. To date, RTP programs at the retail or customer level have been largely experimental, in that they represent only a small fraction of the utility's load. This allows real-time prices to be determined by supply cost forecasts alone, because the load reduction responses that they achieve are not sufficient to change the utility's marginal costs. However, if RTP customers become a significant portion of load, as is hoped by many utilities, this *open loop* approach to pricing will no longer be valid.

This paper provides a methodology for analyzing and planning real-time prices that reflect both utility and customer costs, including the short- and long- term options available to each. A piecewise-linear model proposed for both utility supply costs and customer demand curtailment costs that is consistent with currently available data on these cost structures. This permits a linear programming formulation for minimizing the sum of utility supply costs and customer welfare losses from curtailments. This formulation can also include linear constraints that capture features such as stored hydrosupplies and customer load shifting options. The linear program is shown to satisfy the incentive compatibility and welfare maximizing properties of spot prices. The linear programming optimum may not be achievable when the utility unit commitment decisions and customer curtailment decisions are 0,1 variables. For this case, a greedy algorithm is presented that obtains approximately optimal real-time prices that satisfy the required supply/demand balancing and incentive compatibility properties. The piecewise-linear structure in this paper permits the use of a greedy algorithm for approximate optimization of prices that is more efficient than the Lagrangian relaxation methods used to solve nonlinear unit commitment problems. The algorithm is appropriate for planning real-time prices over medium and long time horizons, but is not intended for optimal short-term dispatching of generation units.

2 Mathematical Description of the Model

It will be assumed that the contract period, which equals the planning horizon, is one year, that real-time prices are to be announced one day in advance, and that prices vary hourly. Over the planning horizon, the model includes uncertainty in both supply and demand through the use of different daily scenarios, $k = 1, 2, \ldots, N$ that can occur with varying frequencies. In practice, the different scenarios would correspond to different weather forecasts, in combination with different availabilities of generating units, different yearly snow pack levels for hydrogeneration, etc. It is assumed that the utility and customers both estimate the same

n_k = the expected number of days of scenario type k during the contract period.

Given that a certain day is of type k, it is assumed that all factors affecting customer load and utility generation are deterministic for the purpose of specifying real-time prices 24 hours ahead. As the day actually progresses, variations from the forecast will, of course, occur (which causes the given prices to be slightly suboptimal), but the corresponding shortfalls or surpluses are simply compensated for by the utility's dispatching procedures, as is currently done for other load variations from forecast. Consequently, pricing related decisions by both the utility and the customer are made on an expected value basis using the best information available at each stage. A piecewise-linear cost function is used for both utility supply cost and customer outage cost. In general, the expected cost per year has the form:

Expected cost per year

$$= \bar{C}_i \bar{X}_i + \sum_k n_k C_{ik} X_{ik} + \sum_{k,t} n_k c_i(t) x_{ik}(t) \tag{1}$$

where $x_{ik}(t) \leq G_{ik} X_{ik}$

The last two terms in (1) are *expected* costs for the contract period, based on the varying fractions of days of each scenario type.

The definitions of the variables and coefficients for the utility are:

$x_{ik}(t)$	=	the energy supplied from unit i on day type k during time period t ;
$c_i(t)$	=	the marginal cost per kWh of energy supplied from unit i during the time period t ;
X_{ik}	=	the fraction of supply type i that is committed on day type k ;
C_{ik}	=	the cost of committing supply type i for day type k ;
$\bar{X}_i$	=	the fraction of supply type i that is committed for the contract period ;
$\bar{C}_i$	=	the fixed cost of committing the maximum possible amount of supply type i for the contract period ;
$G_{ik}(t)$	=	the maximum available energy from supply source i for time period t on day type k ;
S_i	=	the maximum available energy from supply type i for the contract period.

The interpretations of the variables are as follows. Given that a day is forecasted to be of type k, the utility has two levels of decisions to make for each supply source i: The fraction X_{ik} of the supply unit to commit for the day and the level $x_{ik}(t)$ at which to operate in each time period. In many cases X_{ik} is a $0, 1$ commitment variable.Also, for some sources of supply, the utility may be required to make a capacity level decision at the start of the contract period, which is specified by $\bar{X}_i$.For example, for bulk power purchases, a maximum power level needs to be determined, in addition to energy quantities.This supply cost structure is simpler than the one normally considered for unit commitment problems involving thermal and hydrounits.

The parameter S_i is relevant for supply sources such as hydro that operate subject to a fixed storage constraint on total energy. This appears as a constraint.

$$\sum_{k,t}(t) n_k x_{ik}(t) \leq S_i \tag{2}$$

for all i which are constrained on energy.

The definitions of the variables for the customer are:

$x_{ik}(t)$	=	the energy curtailed of load type i on day type k during time period t ;
$c_i(t)$	=	the marginal outage cost per kWh of energy curtailed of load type i during the time period t ;
X_{ik}	=	the fraction of curtailable load type i that is avaliable on day type k ;
C_{ik}	=	the fixed cost of curtailing load type i for day type k ;
$\bar{X}_i$	=	the fraction of load type i that is committed to RTP for the contract period ;
$\bar{C}_i$	=	the fixed cost to commit load type i to RTP for the contract period ;
$G_{ik}(t)$	=	the maximum curtailable load of type i for time period t on day type k ;
S_i	=	the maximum total curtailed energy for load type i for the contract period ;
$L_{ik}(t)$	=	the "normal" load of type i on day k in time period t .

The piecewise-linear costs for the utility and its customers can be used to formulate a linear program for maximizing total welfare, defined as consumer benefit minus supply costs,subject to the production and curtailment constraints. The dual problem yields the optimal real-time prices.The two LPs are as follows.

Primal Problem

$$Min \sum_{i,k,t} n_k c_t(t) x_{ik}(t) + \sum_{i,k} n_k C_{ik} X_{ik} + \sum_i \bar{C}_i \bar{X}_i \tag{3}$$

subject to

$$\sum_i X_{ik}(t) \geq L_k(t) \text{ for all } k,t \tag{4}$$

$$x_{ik}(t) \leq G_{ik} X_{ik}(t) \text{ for all } i,k,t \tag{5}$$

$$X_{ik} \leq \bar{X}_i \text{ for all } k \tag{6}$$

$$\bar{X}_i \leq 1 \text{ for all } i \tag{7}$$

$$\sum_{k,t} n_k x_{ik}(t) \leq S_i \text{ for all } i \tag{8}$$

$$x_{ik}(t), X_{ik}, \bar{X}_i \geq 0 \text{ for all } i,k,t, \tag{9}$$

where $L_k(t) = \Sigma_i L_{ik}(t)$.
Constraint (4) states that all planned load is *served* either through generation or curtailment and (6) states that long-term commitment $\bar{X}_i$ is required before any daily commitment X_{ik} is possible. Constraints (5) and (8) follow from (1) and (2), respectively.

Dual Problem

$$Max \sum_{k,t} L_k(t) n_k \delta_k(t) - \sum_i \bar{\Lambda}_i - \sum_i S_i \mu_i \tag{10}$$

subject to

$$\delta_k(t) - \lambda_{ik}(t) - \mu_i \leq c_i(t) \text{ for all } i,k,t \tag{11}$$

$$\sum_i G_{ik}(t) \lambda_{ik}(t) - \Lambda_{ik} \leq C_{ik} \text{ for all } i,k \tag{12}$$

$$\sum_k n_k \Lambda_{ik} - \bar{\Lambda}_i \leq \bar{C}_i \text{ for all } i \tag{13}$$

$$\delta_k(t), \lambda_{ik}(t), \Lambda_{ik}, \bar{\Lambda}_i \geq 0 \text{ for all } i,k,t \tag{14}$$

Interpretation of the dual variables defines the real time prices and consumer surpluses.The supply and demand constraints (4) imply that the real-time prices are:

$\delta_k(t)$ = the real-time price in time period t for day type k.

At each time t on day k, the customer will curtail available load only if the net surplus $\lambda_{ik}(t)$ derived from (11) is positive, i.e., only if

$$\lambda_{ik}(t) = \{\delta_k(t) - c_i(t) - \mu_i\}^+ > 0 \tag{15}$$

where $\{x\}^+$ denotes the positive part of x. When

$$\lambda_{ik}(t) > 0$$

, the customer will curtail the maximum available load $X_{ik}G_{ik}(t)$, which follows from the complementary slackness condition for (5) and $\lambda_{ik}(t)$. The μ_i in (15) acts as an additive cost component.The μ_i may be the dominant cost for the case in which load shifting is not difficult, i.e., $c_i(t)$ is small. On day type k, the customer will choose to curtail, i.e., set $X_{ik} = 1$, only if the net surplus Λ_{ik} from (12) for curtailing is positive, i.e., only if

$$\Lambda_{ik} = \{\sum_i G_{ik}(t)\lambda_{ik}(t) - C_{ik}\}^+ > 0 \tag{16}$$

Finally, the customer will subscribe to RTP only if the total expected surplus $\bar{\Lambda}_i$ for the contract period is positive, i.e.,

$$\bar{\lambda}_i = \{\sum_k n_k\Lambda_{ik} - \bar{C}_i\}^- > 0 \tag{17}$$

Available supply is provided at time t and day k only if real-time price exceeds marginal cost $c_i(t) + \mu_i$, that supply is committed on day k only if the sum producer surplus exceeds commitment cost C_{ik}, and commitment for the contract period occurs only if the expected net producer surplus exceeds the fixed commitment cost.

Deciding between RTP and Another Service Option

The customer's decision to subscribe to RTP is typically a choice between RTP and one or more other options.The linear programming formulation can be generalized to incorporate this decision as shown below. Suppose that the choice is between RTP and a service option that has a fixed cost B_i for the contract period and offers prices $p_k(t)$, which may or may not vary with k and t. The customer's choice is made based on which option provides the greater consumer surplus. As noted in the LP formulation, consumer surplus is maximized when the customer charges plus interruption costs are minimized.It is assumed that the prices $p_k(t)$ are fixed, which allows the corresponding customer curtailments $y_{ik}(t)$ to be determined in advance. Treating $L_{ik}(t)$ as the potential load available, the total charges plus interruption costs associated with the alternative service option are given by

$$Z_i = \sum_{k,i}[L_{ik}(t) - y_{ik}(t)]p_k(t) + \sum_{k,i} n_k y_{ik}(t)c_i(t) + Bi \qquad (18)$$

which is treated as a constant in the LP for determining the prices $\delta_k(t)$. Customer i will select the RTP option only if the charges plus interruption costs associated with the optimal $\delta_k(t)$ are less than or equal to Z_i. Based on the interpretations of the dual variables $\bar{\Lambda}_i$ and Λ_{ik} in (15), (16), and (17) and constraints (11), (12) and (13), it can be seen that $\bar{\lambda}_i \geq$ cost savings–interruption cost for customer i, with equality holding at the optimal solution. Thus,

$$\sum_{k,i} n_k L_{ik}(t)\delta_k(t) - \bar{\Lambda}_i \leq \text{customer i's charges + interruption costs} \qquad (19)$$

with equality holding at the optimal solution. Therefore, incorporating the constraint

$$\sum_{k,i} n_k L_{ik}(t)\delta_k(t) - \bar{\Lambda}_i \leq Z_i \qquad (20)$$

in the dual objective acts as an additional constraint on the revenue that is generated by the prices $\{\delta_k(t)\}$. This generates an additional primal variable $\bar{Y}_i$, defined as the fraction of load that customer i allocates to the alternative service option. The corresponding modification of the primal problem is to add a sum of terms $Z_i\bar{Y}_i$ to the objective function and to modify (4) and (7) as

$$\sum_i [x_{ik}(t) + L_{ik}(t)\bar{Y}_i] \geq L_k(t) \text{ for all k,t} \tag{21}$$

$$\bar{X}_i + \bar{Y}_i \leq 1 \text{ for all i} \tag{22}$$

It can be seen that setting $\bar{Y}_i$ to 1 removes all customer i's load from the supply and demand constraint (4) and substitutes different costs in the objective function.

3 Discussion

Economic analyses have demonstrated that electric utilities can achieve greater efficiency by offering a diversity of service options to their customers. Real-time pricing, which approximates spot pricing and announces prices in advance to provide sufficient time for response, holds great potential for efficiency improvements, if used on a broader scale. In order to expand RTP beyond its current limited status, a comprehensive methodology is needed that analyzes its impact on customer consumption and commitment decisions. This paper develops a model for analyzing RTP that reflects fixed and variable costs for both the utility and its customers and which minimizes the sum of supply and service curtailment costs. The model can be optimized as a linear program over both short and long time horizons. An approximate, mixed integer algorithm provides an efficient method for solving the case of piecewise-linear costs in which capacity commitments and customer curtailments are 0,1 decision variables. Since RTP is one of several available service options, it is important to be able to analyze the customer's decision to subscribe. This model can be applied to include the prices and terms of other available service options in setting real-time prices and analyzing subscription decisions.

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Optimization Of Nonlinear Biotechnological Processes With Linear Programming: Application To Citric Acid Production By Aspergillus Niger

by B. Űrek and O. Pembe

1 Verbal description of the model

The metabolic pathway and the properties of many enzymes involved in the citric acid biosynthesis in the mold Aspergillus niger are well known.This fact together with the availability of new theoretical frameworks aimed at quantitative analysis of control and dynamics n metabolic systems,has allowed us to construct a mathematical model of the carbohydrate metabolism in Aspergillus niger under conditions of citric acid accumulation.The model makes use of the S-system representation of biochemical systems,which renders it possible to use linear programming to optimize the process.It was found that maintaining the metabolite pools within narrow physiological limits(20% around the basal steady-state level)and allowing the enzyme concentrations to vary within a range 0.1 to 50 times their basal values.It is possible to triple the glycolytic flux while maintaining 100% yield of substrate transformation.To achieve these improvements it is necessary to modulate seven or more enzymes simultaneously.Although this seems difficult to implement at present,the results are useful because they indicate what the theoretical limits are and because they suggest several alternative strategies.

Citric acid is a widespread chemical that is extensively used in numerous industrial processes,ranging from food technology to various branches within the chemical industry.Most of its world production is obtained by fermentation of low quality sugars by using the mold Aspergillus niger.

Recent developments in molecular biology make it possible to manipulate genes for particular enzymes,and the availability of enough information about the biochemistry of these processes.Therefore it is becoming more feasible to construct reliable mathematical models of intact metabolic systems.Construction of such models constitutes the basis for systematic strategies to improve biotechnological processes.It provides guidance in the determination of which enzymes should be overexpressed,and by how much,to obtain the maximum yield in a desired biological transformation,while,at the same time,guaranteeing cell viability.*Sensitivity analysis*[1,2] which pinpoints enzymes that have strong control over the desired response helps to answer this question,at least to some degree.However,information provided by sensitivity analysis has proven to be insufficient.

Some authors have made attempts to apply optimization criteria to metabolic *systems*[3,4,5] and individual enzymes.But given the intrinsic nonlinear character of most metabolic systems,results are scarce and confined to a small number of variables.This situation is in stark contrast to the linear domain,where many mathematical procedures are available that straightforwardly execute searches of optimum solutions within given constraints.A natural bridge between the two domains is given by biochemical system theory (BST)[6,7,8]. S-systems models that derive from BST are nonlinear and able to represent all types of dynamic behaviors.However,their steady states are characterized by linear equations that can be evaluated both analytically and numerically.This combination of nonlinear dynamics with linear steady state characteristics offers a true alternative to optimization questions in biotechnology.

The goal of this paper is to provide methodologies that offer assistance in the search of the profiles of enzyme activities that yield maximal citric acid production flux in Aspergillus niger.

Before constructing the mathematical model we had better define the S-system which is used in modeling the carbohydrate degradation and oxalacetate formation in Aspergillus niger under conditions of citric acid accumulation in our model.

S-SYSTEMS: S-systems originally were developed for analysis of organizationally complex systems such as cellular and molecular networks in biology.The name S-system stands for synergistic and saturable system.Each equation of an S-system describes the change in an elementX_i with time and is composed of two terms,one for net production,representing processes that contribute to the increase in X_i,and the other for net degradation representing processes that contribute to the decrease in X_i.An element may be a molecule, cell, organism , or any other quantifiable characteristic of a system.

Mathematically,the net production and net degradation terms for each element of an S-system are products of power-law functions.Thus,S-systems have the following form:

$$X_i^1 = \alpha_i \Pi_{j=1}^n X_j^{g_{ij}} - \beta_i \Pi_{j=1}^n X_j^{h_{ij}} \qquad i = 1, \ldots, n. \tag{1}$$

where X_i^1 is the change in X_i with time,α_i and β_i are multiplicative parameters or apparent rate constants for net production and net degradation,g_{ij} and h_{ij} are exponential parameters or kinetic orders for the influence X_j

has on net and net degradation of X_i,and n is the number of variables in the system equal to the number of differential equations if system is assumed for simplicity to be autonomous.Values for the X_i's are real positive,for the α_i's and β_i's real nonnegative,and for the g_{ij}'s and h_{ij}'s real.

S-systems have been applied successfully in the analysis of biochemical pathways,genetic circuits,immune networks,patterns of growth and development,and ecological interactions.

2 Mathematical Description of the Model

The BST approach was used *recently*[9] to design a detailed quantitative S-system model of carbohydrate degradation and oxalacetate formation in Aspergillus niger,under conditions of citric acid accumulation.The model addressed the citric acid production stage,when the system attains a well characterized steady state in which citrate production is the only activity of quantitative importance.

In the S-system model,each net rate law for synthesis and degradation is represented by a product of power-law functions of all variables that have an influence upon the net rate law in question.The rate of a given processV_i is written as:

$$V_i = \alpha \Pi_{j=1}^{n+m} X_j^{g_{ij}} \tag{2}$$

where X_j are variables modeling enzymes,metabolites,or effectors,that affect the rate in question.The indices $j = 1, \ldots, n.$ refer to dependent variables,while $j = n+1, \ldots, n+m.$ refer to independent variables.Those processes that characterize reactions forming a metabolite X_i are aggregated to give single rate law for net synthesis,V_i^+.Similarly,those rate laws that characterize reactions removing a metabolite X_i are aggregated to give single rate law for net degradation,V_i^-.The descriptive equations for a biochemical system can then be written in terms of power-law functions as follows:

$$\frac{dX_i}{dt} = \alpha_i \Pi_{j=1}^{n+m} X_j^{g_{ij}} - \beta_i \Pi_{j=1}^{n+m} X_j^{h_{ij}} , \qquad i = 1, \ldots, n \qquad , \tag{3}$$

The parameters α_i and g_{ij} are the rate constants and kinetic orders associated with the rate law for net synthesis of X_i.Similarly β_i and h_{ij} are associated with the rate law for net degradation of X_i.The kinetic order parameters g_{ij} and h_{ij} are defined as:

$$g_{ij} = \frac{(\partial V_i^+)}{(\partial X_j)} \cdot \frac{(X_j)}{(V_i^+)_0} \qquad \%, \tag{4}$$

$$h_{ij} = \frac{(\partial V_i)}{(\partial X_j)} . \frac{(X_j)}{(V_i)_0} \tag{5}$$

where the additional index 0 indicates that a quantity is measured at the steady state level of metabolit concentration.Values for the parameters α_i and β_i are determined in such a way that the modeled rate and the power-law approximation are equivalent at steady state.

Numerical reference system:

$$\frac{dX_1}{dt} = 0.459 X_8^{0.23} X^{0.0009} X_{10} - 1.83310^{-3} X_1^{2.194} X_2^{-1.333} X_{11}^{0.2} X_{12}^{0.8} \tag{6a}$$

$$\frac{dX_2}{dt} = 3.2510^{-4} X_1^{2.665} X - 1667_2 X_{12} - 1.861 X_2^{0.9} X_8^{0.056} X_{13} X_{21}^{0.65} X_{22}^{-0.3} \tag{6b}$$

$$\frac{dX_3}{dt} = 1.861 X_2^{0.9} X_8^{0.056} X_{13} X_{21}^{0.65} X_{22}^{-0.3} - 0.21 X_3^{0.9} X_7^{-0.028} X_8^{-3.196} X_{14} \tag{6c}$$

$$\frac{dX_4}{dt} = 0.42 X_3^{0.9} X_7^{-0.028} X_8^{-3.196} X_{14} - 2.538 X_4^{0.836} X_8^{0.161} X_{15}^{0.509} X_{20}^{0.486} \tag{6d}$$

$$\frac{dX_5}{dt} = 1.895 X_4^{0.7} X_8^{0.315} X_{15} - 4.2310^{14} X_5^{3.2} X_6^{-2.2} X_7^{3.788} X_{16}^{0.945} X_{17}^{0.054} \tag{6e}$$

$$\frac{dX_6}{dt} = 2.0410^{15} X_5^{3.333} X_6^{-2.33} X_7^{4.005} X_{16} - 0.005 X_6^{0.813} X_{18}^{0.979} X_{19}^{0.0204} \tag{6f}$$

$$\frac{dX_7}{dt} = 0.204 X_3^{0.873} X_7^{-0.028} X_8^{-4.06} X_{14} - 8.68710^6 X_5^{1.618} X_6^{-1.136} X_7^{2.203} X_{16}^{0.486} X_{23}^{0.513} \tag{6g}$$

$$\frac{dX_8}{dt} = 0.841 X_3^{0.9} X_7^{-0.028} X_8^{-3.196} X_{14}$$

$$-0.308 X_2^{0.225} X_4^{0.179} X_8^{0.257} X_9^{0.294} X_{10}^{0.312} X_{13}^{0.25} X_{15}^{0.256} X_{21}^{0.162} X_{22}^{-0.075} X_{24}^{0.18} \tag{6h}$$

The S-system model allows us to gain insight in the steady state and dynamic features of the carbohydrate metabolism in Aspergillus niger.The goal of this analysis is to optimize citric acid production at steady state.In other words the goal is to move the original steady state system to a new steady state that satisfies certain boundary conditions.But in this optimization there are some restrictions because the S-system model and some of the physiological and logistic constraints are nonlinear.However,the steady state equations are linear when the variables are are expressed in logarithmic coordinates,and because a function and its logarithm assume their maxima for the same argument,yields or fluxes can be optimized with linear programs expressedd in terms of the logarithms of the original variables.Also,typical constraints that the optimized system has to satisfy reduced to linear equations in a logarithmic coordinate system. For instance,the flux between two pools is represented in the S-system model by a product of power-law functions,and a constraint requiring that this flux be within a given physiological range reduces to two linear equations in logarithmic coordinates.Similarly the equality of two fluxes becomes linear and so does the constraint that the concentration of an intermediate should not exceed a certain physiologically tolerable threshold.Transporting the steady state system and the constraints in to a logarithmic space thus reduces the nonlinear optimization problem to a problem of straightforward linear *programming*.[10]

The problem of optimizing a particular flux in an S-system under typical constraints may thus read as follows:

LINEAR PROGRAM

1. maximize ln(flux)

subject to

2. steady state equations,expressed in logarithms of variables
3. ln(dependent or independent variable)$\leq$ constant
4. ln(dependent or independent variable)$\geq$ constant
5. ln(dependent or independent variable)= constant
6. ln(variable) unrestricted
7. ln(flux)$\leq$ constant
8. ln(flux)$\geq$ constant
9. ln(flux) unrestricted
10. ln(flux 1)-ln(flux 2)=0

In this formulation,(1) is the objective function that is linear in the logarithms of the involved dependent and independent variables.(2) assures that the optimized system is in a steady state,no matter what the altered enzyme concentrations are.Both (3) and (4) constrain variables to stay within certain limits.(5) forces the variable to be at a given value whereas (6) is an option that permits any real value for the logarithm of a variable and thus any positive real value for the variable itself.(7) through (9) are the corresponding constraints on fluxes,and (10) ensures that flux 1 and flux 2 are equivalent.

So far we have explained the use of linear program in the S-system model.One of the application of this S-system model using linear programming is the citric acid production by Aspergillus niger.As a first exploration of optimal enzyme profiles,we assume that all enzymes can be modulated.This assumption implies maximum control over the system and is expected to yield the theoretically highest possible citric acid production under physiological conditions.

MAXIMIZATION OF THE GLYCOLYTIC FLUX(UNCONSTRAINED GLYCOLYTIC SOLUTION):The first step in setting up a linear program is the definition of a proper objective function.Because citric acid production is directly related to the level of glycolytic $flux^{11}$ a reasonable optimization might use as the objective function the flux through the pyruvate kinase reaction,V_4^+.We call the solution to this optimization problem the unconstrained glycolytic solution.The objective function for this linear program in logarithmic coordinates reads:[Eq.(6d)]

maximize

$$ln(0.42X_3^{0.9}X_7^{-0.028}X_8^{-3.196}X_{14}-2.538X_4^{0.836}X_8^{0.161}X_{15}^{0.509}X_{20}^{0.486}) \quad (7)$$

The second step is the formulation of the steady state constraints,again expressed in terms of the logarithms of the variables.

subject to

$$ln(0.459X_8^{0.23}X_9^{0.0009}X_{10}-1.83310^{-3}X_1^{2.194}X_2^{-1.333}X_{11}^{0.2}X_{12}^{0.8})$$

$$ln(3.2510^{-3}X_1^{2.665}X_2^{-1.667}X_{12}-1.861X_2^{0.9}X_8^{0.056}X_{13}X_{21}^{0.65}X_{22}^{-0.3})$$

$$ln(1.861X_2^{0.9}X_8^{0.056}X_{13}X_{21}^{0.65}X_{22}^{-0.3}-0.21X_3^{0.9}X_7^{-0.028}X_8^{-3.196}X_{14})$$

$$ln(1.895X_4^{0.7}X_8^{0.315}X_{15}-4.2310^{14}X_5^{3.2}X_6^{-2.2}X_7^{3.788}X_{16}^{0.945}X_{17}^{0.054})$$

$$ln(2.0410^{15}X_5^{3.333}X_6^{-2.33}X_7^{4.005}X_{16}-0.005X_6^{0.813}X_{18}^{0.979}X_{19}^{0.0204})$$

$$ln(0.204X_3^{0.873}X_7^{-0.028}X_8^{-4.06}X_{14}-8.68710^6X_5^{1.618}X_6^{-1.136}X_7^{2.203}X_{16}^{0.486}X_{23}^{0.513})$$

$$ln(0.841X_3^{0.9}X_7^{-0.028}X_8^{-3.196}X_{14}$$
$$-0.308X_2^{0.225}X_4^{0.179}X_8^{0.257}X_9^{0.294}X_{10}^{0.312}X_{13}^{0.25}X_{15}^{0.256}X_{21}^{0.162}X_{22}^{-0.075}X_{24}^{0.18})$$

In this linear program just the objective function and the steady state constraints are expressed but there are logarithmic functions in the above part of this program so we should express whole program as standard LP form as follows:In this program $y_i = ln(X_i)$

maximize

$$0.9y_3-0.028y_7-3.196y_9+y_{14}$$

subject to

Steady State Constraints:

$$-2.194y_1+1.333y_2+0.23y_8+0.0009y_9+y_{10}-0.2y_{11}-0.8y_{12} = -5.523$$

$$2.665y_1-2.567y_2-0.056y_8+y_{12}-y_{13}-0.65y_{21}+0.3y_{22} = 8.651$$

$$0.9y_2-0.9y_3+0.028y_7+3.252y_8+y_{13}-y_{14}+0.65y_{21}-0.3y_{22} = -2.179$$

$$0.9y_3-0.836y_4-0.028y_7-3.357y_8+y_{14}-0.509y_{15}-0.486y_{20} = 1.797$$

$$0.7y_4-3.2y_5+2.2y_6-3.788y_7+0.315y_8+y_{15}-0.945y_{16}-0.054y_{17} = 33.039$$

$$3.333y_5-3.143y_6+4.005y_7+y_{16}-0.979y_{18}-0.0204y_{19} = -40.447$$

$$0.873y_3-1.618y_5+1.132y_6-2.231y_7-4.06y_8+y_{14}-0.486y_{16}-0.513y_{23} = 17.563$$

$$-0.225y_2+0.9y_3-0.179y_4-0.028y_7-3.453y_8-0.294y_9-0.312y_{10}$$

$$-0.25y_{13}+y_{14}-0.256y_{15}-0.162y_{21}+0.075y_{22}-0.018y_{24} = -1.003$$

Constraints On Decision Variables:

These are the enzyme concentrations and we must determine which enzyme concentrations will vary and how much.Allowing all enzymes to change between 0.1 and 50 times their baseline levels we have intervals of the decision variables as follows.But the only exceptions to this limiting are (X_9),(X_{11}),(X_{17}),(X_{19}),(X_{21}) and(X_{22}).Because these enzymes divert flux from the pathway leading to citrate and their concentrations should be maintained constant at the basal values.So our constraints on decision variables are:

$$y_9 = 5.669$$

$$-6.437 \leq y_{10} \leq -0.223$$

$$y_{11} = -4.074$$

$$0.425 \leq y_{12} \leq 6.639$$

$$-4.893 \leq y_{13} \leq 1.321$$

$$-4.120 \leq y_{14} \leq 2.015$$

$$-6.502 \leq y_{15} \leq -0.287$$

$$-0.580 \leq y_{16} \leq 5.634$$

$$y_{17} = -3.036$$

$$-2.919 \leq y_{18} \leq 0.886$$

$$y_{19} = -6.214$$

$$-5.404 \leq y_{20} \leq 0.811$$

$$y_{21} = 0.854$$

$$y_{22} = 1.726$$

$$-2.302 \leq y_{23} \leq 3.912$$

$$-2.302 \leq y_{24} \leq 3.912$$

Constraints On The Dependent Variables:

As with the decision variables it is reasonable to limit the range of variation of the dependent variables which represent the metabolite concentrations.Because overexpression of enzymes associated with the overproduction of a given product,evokes changes in the intermediate metabolite pools which have kinetic effects on the rest of the metabolic system.These chains of effects usually cause losses in growth rate or fitness and subsequently lead to poor yield in product synthesis.So we allow changes of up to 20% around the steady state levels.The constraints on the dependent variables are the following:

$$-2.525 \leq y_1 \leq -2.120$$

$$-3.912 \leq y_2 \leq -3.507$$

$$-4.828 \leq y_3 \leq -4.423$$

$$-2.343 \leq y_4 \leq -1.938$$

$$-8.369 \leq y_5 \leq -7.963$$

$$0.524 \leq y_6 \leq 0.929$$

$$-3.576 \leq y_7 \leq -3.170$$

$$-0.916 \leq y_8 \leq -0.511$$

FLUX CONSTRAINED MAXIMIZATION OF THE GLYCOLYTIC FLUX (CONSTRAINED GLYCOLYTIC SOLUTION):The second step in setting up this LP is the constrained maximization of the glycolytic flux.Because in the optimization process above the system deviates rather far from the original steady state operating point that underlies the S-system model,so some of the stoichiometric relationship that hold in the basal solution are no longer satisfied.To avoid this we add a restriction that reflects the required dependency between fluxes.So the flux $V_{3,4}$ must be less than the input flux,V_1:

$$\frac{V_{3,4}}{2V_1^+} < 1$$

which is equivalent to:

$$\frac{V_1^+}{V_4^+} > 0.5$$

From equation (6a) and (6d) the power-law expressions for V_1^+ and V_4^+ are obtained.After taking the logarithm and rearranging we obtain:

$$-0.9y_3+0.028y_7+3.426y_8+0.00093y_9+y_{10}-y_{14} > -0.780 \tag{8}$$

And we included this constraint into steady state constraints as the flux stoichiometric constraint.And we call the solution to this linear program the constrained glycolytic solution.

MAXIMIZATION OF CITRATE PRECURSORS(STOICHIOMETRIC SOLUTION): The third step in setting up this LP is the maximization of the citrate precursors.In the first step we explained that the citric acid production is directly related to the level of glycolytic flux.Even though there is a direct correlation between the glycolytic flux and the citrate synthesis a more accurate criterion of productivity is the maximization of the reactions that synthesize the precursors of citrate namely (V_4) and (V_6).Both precursor expressed in terms of the power-law function and they take the form:

$$V_4 = \alpha_4^{'}.X_4^{g_{4.4}}.X_{20}^{g_{4.20}}$$

$$V_6 = \alpha_6^{'}.X_6^{g_{6.6}}.X_{18}^{g_{6.18}}$$

if we write these equation explicitly:

$$V_4 = 0.8697.X_4^{0.98}.X_{20} \tag{9a}$$

$$V_6 = 0.0047.X_6^{0.83}.X_{18} \tag{9b}$$

In our model the citrate synthase reaction,consumes a molecule of V_4 and a molecule of V_6 to produce a molecule of citrate,and impose the additional constraint that these two rates should be equal:

$$V_4 = V_6$$

After taking the logarithms in equation (9a),(9b) and rearranging terms, we obtain :

$$0.98y_4-0.83y_6-y_{18}y_{20} = 5.2031 \tag{10}$$

and we add this stoichiometric constraint into steady state constraints.

To optimize the system under these conditions we thus maximize:

$$0.98y_4+y_{20} \tag{11}$$

And we add the last constraint Eq (8) of the constrained glycolytic maximization to the last LP model.And we call this solution to the linear program the stoichiometric solution.

3 Discussion

Variable	*Basal*	*Unconst. glyc.*		*Const. glyc.*		*Stoichiometric*	
Dpndt	X_{i_B}	X_i	X_i/X_{i_B}	X_i	X_i/X_{i_B}	X_i	X_i/X_{i_B}
X_1	0.10	0.12	1.2000	0.12	1.200	0.12	1.20
X_2	0.025	0.02	0.8	0.02	0.8	0.02	0.8
X_3	0.01	0.01	1.0	0.008	0.8	0.008	0.8
X_4	0.12	0.096	0.8	0.096	0.8	0.096	0.8
X_5	$2.9e^{-4}$	$3.48e^{-4}$	1.2	$3.48e^{-4}$	1.2	$2.32e^{-4}$	0.8
X_6	2.11	2.53	1.2	2.53	1.2	1.68	0.8
X_7	0.035	0.034	0.97	0.028	0.8	0.042	1.2
X_8	0.5	0.6	1.2	0.6	1.2	0.4	0.8
Indpndt	X_{i_B}	X_i	X_i/X_{i_B}	X_i	X_i/X_{i_B}	X_i	X_i/X_{i_B}
X_{10}	0.016	0.23	14.6	0.039	2.47	0.043	2.71
X_{12}	15.3	192.77	12.6	20.94	1.37	20.94	1.36
X_{13}	0.075	2.69	36	0.29	3.9	0.3	4.0
X_{14}	0.15	7.5	50	1.05	7.02	0.29	1.94
X_{15}	0.015	0.72	48.1	0.0096	0.64	0.058	3.91
X_{16}	5.6	280	50	6.34	1.13	11.38	2.032
X_{18}	0.54	27	50	0.25	0.47	2.25	4.16
X_{20}	0.045	1.15	25.7	1.10	24.6	0.19	4.22
X_{23}	1	12.68	12.68	14.20	14.20	4.16	4.16
X_{24}	1	50	50	50	50	5.57	5.57
Fluxes	$(V_{ij})_b$	V_{ij}	$V_{ij}/(V_{ij})_b$	V_{ij}	$V_{ij}/(V_{ij})_b$	V_{ij}	$V_{ij}/(V_{ij})_b$
V_1^+	0.00630	0.09500	15.0700	0.04600	2.53000	0.01600	2.54000
V_1	0.00126	0.00133	1.05000	0.00133	1.05	0.00133	1.05
V_4^+	0.01	0.3	30	0.0325	3.25	0.0325	3.25
V_4	0.0049	0.10	20.40	0.096	19.59	0.016	3.26
V_6	0.0048	0.279	58.125	0.0026	0.54	0.016	3.33
V_{FM}	0.0001	0.0001	1	0.0001	1	0.0001	1
V_{GOT}	0.00028	0.00033	1.17	0.00033	1.17	0.00022	0.78
Yield	76.1%	105.2%		16.25%		100%	

$Where\ X_{i_B}\ :\ X_{i_{Basal}}$

$RESULTS:$

Inspection of the flux values shows that in this solution the citrate production (V_1) is considerably higher than in the original system : In the basal solution ,the rate of citrate production is 0.0049μ mol/min mg dry weight, whereas in the unconstrained glycolytic solution this value is increased to

0.1μ mol/min mg dry weight ,which corresponds to an improvement of 20.40 times.At the same time , the mitochondrial malate synthesis rate, V_6 is 0.279μ mol/min mg dry weight ,which is 2.79 times greater than the mitochondrial pyruvate synthesis rate, V_4. This increase would cause an excess of mitochondrial malate production that could not be entirely processed by citrate synthesis.

if we examine the above table we will see that the input flux , V_1 increases 2.53-fold , while the rate of citrate production , V_6 is about half the basal rate .This can be interpreted as a situation in which the yield of substrate transformation is inferior (16.25% vs. 76.1%) and most of the subcitrate is converted into malate.Although the constrained glycolytic solution does not deviate significantly from the mass conservation constraints , it represents a solution that is inferior to the basal solution by being very example in term of wasteful consumption of substrate and poor yield.

Undesired features are prevented in the stoichiometric solution.In this solution,the glucose uptake rate (V_6;0.016μ mol/min mg dry weight) production is 3.33 times higher than the basal solution and 6.15-fold higher than in the constrained glycolytic solution.Moreover , the final balance shows that all substrate is converted into citrate.Thus at the steady state of the stoichiometric solution the organism consumes more glucose and produces citrate at a higher rate and with a better yield than under the original,basal conditions.

There are important differences between the enzyme profiles between the enzyme profiles of the different solutions.First, the unconstrained glycolytic solution requires between 12 and 50-fold more activity than the basal solution,while both the constrained glycolytic and the stoichiometric solution show much lower amplifications of enzyme activities.In the constrained glycolytic solution only one enzyme (X_{21}) is amplified 50-fold and a second one 24-fold (X_{20}), while most of the remaining amplification factors are below 10.The stoichiometric solution is even more efficient:The highest amplification factor is 5.57 (X_{21}),whereas the others range between 1 and 5.That means:first,no drastic changes are needed to increase citrate production by more than three times;and,second,both glycolytic solutions are more expensive than the stoichiometric solution,because they both involve higher rates of gene expression and protein synthesis.We can conclude that the stoichiometric solution produces citrate at a rate more than three fold greater than the basal system.Furthermore,it consumes more substrate but does not require significantly higher protein synthesis than the basal solution.The yield of substrate transformation is 100% in the stoichiometric solution,and is thus higher than in the basal solution.

COMMENTS:

The proposed method of optimization consists of three components.The first one is a reliable data base.This component has been the target of biotechnology since its beginnings.The second component of the proposed optimization method is a mathematical approach and,in particular the representation of a biochemical process in a suitable form.As an especially convenient biochemical model we use S-system differential equations that derive from biochemical systems theory.S-systems are unique in that they have a scope wide enough to capture all relevant nonlinearities but that they,at the same time,reduce the linear equations when one is concerned with the steady state characteristics.The third component of the proposed method is the solution of linear programs.Application of the constraint linear optimization method to the production of citric acid by Aspergillus niger implies that the metabolic pathway leading to the accumulation of citric acid.This is evident in the rate of citric acid production and in the yield of glucose transformation into citric acid.While some of the results on citric acid production are not surprising,others are interesting in themselves,and they also indicate the potential of the proposed method.First,the results demonstrate that the method is working and produces reasonable results.This model is a quantitative tool that integrates all available information in a logical fashion and provides a reasonable ranking of the importance of enzymes for optimizing processes of biotechnological relevance.

LIMITATIONS:

Alternative approaches that satisfy all stoichiometric relationships often indicate serious problems in model implementation and evaluation or show greater inaccuracies in metabolite *levels*.[12,13] In this model at least seven enzymes would have to be altered simultaneously and in particular quantities if citric acid production were to increase to a significant degree.Also there is no absolute guarantee that the suggested enzyme modifications with qualitatively and quantitatively produce the predicted results.Assumptions in setting up the model,uncertainties in the kinetic data used for parameter estimation,and effectors that for simplicity are considered constant could affect the altered system.

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The Disentanglement of the Bankruptcies of al-Manakh Stock Market Crash via Linear Programming

by H. Yıldız and G. Yıldız

1 Verbal Description of the Model

Kuwait is a small country whose economy is heavily dependent on its considerable oil wealth. With the government owning almost half the countries share-holding companies, small and medium size Kuwaiti investors are left with few alternatives. When the oil revenues of government increased during the third oil price hike in 1979, public spending increased significantly. This caused funds to flow toward stocks, which caused to create an unofficial stock market called Souk al-Manakh. In that market, institutional regulations and government control were absent which made it a highly speculative market. In the presence of excess liquidity and the heavy use of postdated checks, stock prices started to rise sharply. Consequently, a strong demand for new issues appeared. Within 18 months more than 80 new companies were established, as it turned out, quite a few were established for speculative purposes only. About 6000 individuals and corporations were trading in this market by the summer of 1982.

The stock exchanges were made without any transfer of cash, but with high premium rated, often with 60 to 200 percent, postdated checks. A trader who buys in this way was relying on the fact that a 100 percent price increase in a few weeks was considered as normal. No supervision by any regulatory body such as the Central Bank was existed, and this anarchic market crashed when one large trader defaulted in August 1982. This collapse initiated a chain of other collapses of traders. At the end, traders were left with postdated checks and stocks of different companies. Most of the traders have both receivables from and payables to other traders. The crisis involved individual speculators, legitimate banks, industrial firms, and commercial companies. The outstanding debt amounted to \$94 billion, which was the face value of al-Manakh's 29000 postdated checks. This collapse showed its impact on the rest of the economy, securities traded in al-Manakh lost 60 to 98 percent of their peak values, real estate values dropped by about 40 to 60 percent from their pre-cash highs, 350 bankruptcies were declared during the early days of the crisis.

The government set up a task force, headed by the Minister of Finance, to resolve the crisis. However, resolving the crisis was a difficult task since no financial records of outstanding trade accounts among traders existed at the time of the crash. Even if they were existed, it would have been still impossible to resolve the entanglements one at a time because of the insolvent's simultaneous dependencies on each other and the "Domino Effect".

The main objective was to compute the equilibrium debt settlement ratio (DSR) for each trader, that is, how each trader would divide his assets fairly to all his debtors. Three Operations Research experts explored the possibility of using a simultaneous equation systems. However the solution of the linear equations yielded an unfair result for solvents. If a trader was solvent even if he can not get his receivables, then the system of linear equations would provide a DSR greater than one, which means the trader should pay more than he owed. Consequently, DSRs of other traders were raised, and falsely helped debtors who are borderline insolvents to become solvents. So to compute the DSRs, linear programming is applied, which would provide the flexibility of applying bounds on the DSR, namely $0 \leq DSR \leq 1$. The approach was to compute the DSRs in a way that protected the rights of all creditors while maintaining the utmost level of fairness in asset distribution. Several objective functions was considered for this purpose. Several objective functions was considered for this purpose such as "Minimizing number of bankrupticies", "Maximizing total payments to all creditors", "Maximizing the sum of insolvent's DSRs", etc. The minister selected the latter one as to be used in the linear programming. The reason was that if all traders paid their debts in full, the debt settlement problem would disappear. In addition this objective function would serve to minimize the negative impact on the economy and to guarantee that all traders were treated equally.

At the outset, the al-Manakh stock market traders are classified into four groups:

1- Definitely solvent traders (I1) have assets exceeding their payables and can therefore pay all their debts in full even if they are not paid their receivables.

2- Seemingly solvent traders (I2) have assets and receivables that together exceed their payables; however, some of their debtors may default on a portion of their debts, rendering the sum of assets and actual receivables less than payables.

3- Definitely insolvent traders with positive receivables (I3) have payables that exceed the sum of their assets plus receivables.

4- Definitely insolvent traders with zero receivables (I4) have payables that exceed their assets and no receivables.

During the conceptualization step two important assumptions were made: i- Each insolvent trader must treat his creditors equally by paying each of them the same percentage on each Kuwaiti Dinar worth of his debt, ii- For the time being all asset were assumed to be homogeneous, perfectly substitutable and of identical risk. It is also noticed that the vast majority of al-Manakh traders were small definitely solvent traders. For the sake of simplicity, these are excluded from the model, since they were able to pay all their debts. Definitely insolvent traders with zero total receivables are also excluded since they have no receivables. The problem formulation was therefore reduced to the identification of solvent from insolvent traders within the relatively small number of seemingly solvent traders given by the set I2, in addition to determining the debt settlement ratios for traders in the two sets, I2 and I3, whose assets are used for debt settlement. The payments made by traders in I1 and I4 were computed in a separate operation outside the LP model and added to the assets of traders in the set I2 and I3.

After computing the DSRs for each trader, there remains another problem. Since each trader's assets consisted typically of different types (e.g. stocks, bonds, cash, real estate, intangibles, etc.) with different degrees of financial risks, in which combination of financial instruments should debts be paid to each creditor? It is only fair to distribute traders' assets to their creditors in the same proportion as that of their asset portfolios. Deciding this was very complicated because some assets would accrue to traders from other traders, without knowing neither the quality nor the form a priori. Thus, a system of simultaneous linear equations is formulated to resolve entanglements among all traders for each type of asset. Latter the lp model is extended to include this system of linear equations.

The mathematical models here provided the grounds on which court decisions were made and also provided the basis for the government's final resolution of the crisis, which involved financial compensatory schemes for al-Manakh creditors and repayment schedules for debtors. The models proved to be effective, equitable, and robust. Without this work, courts would have been overwhelmed for years with criminal, commercial and bankruptcy cases resulting from 29000 postdated checks; it avoided more than 10 billion in court costs and attorney fees.

2 Mathematical Description of the Model

Decision Variables

λ_i = The fraction of payments made by trader i to all his debtors, (by definition $0 \leq \lambda_i \leq 1$).

λ_{ik} = The portion of asset type k trader i uses to settle his debt.

Parameters

a_i = total assets of trader i, Kuwaiti Dinars (KD):

$$a_i = q_i + \sum_{j \in I_1} r_{ij} + \sum_{j \in I_4} \lambda_j r_{ij}, i \in I_2 \cup I_3$$

p_i = total payables of trader i, KD.

C_i = objective function coefficient or weight of trader i debt settlement ratio.

r_{ij} = receivables of trader i due from trader j, KD.

A_k = a vector of asset type k for all traders, KD (k = 1,2,...,K)

A = a vector of total assets, KD, where $A = \sum_{k=1}^{K} A_k$

a_{ik} = the amount of asset type k that belongs to trader i, KD.

m = number of different types of assets.

$\tilde{\lambda}$ = vector of debt settlement ratios for all traders.

$\tilde{\lambda}_k$ = vector of asset type k portion used for debt settlement ratios for all traders.

β_k = parameter indicating the priority of using up each type of asset for solvent traders.

N = total number of traders.

D = matrix of all traders payables and receivables, where D is defined by as follows:

$$D = \begin{pmatrix} p_1 & -r_{12} & -r_{13} & \dots & -r_{1n} \\ -r_{21} & p_2 & -r_{23} & \dots & -r_{2n} \\ \dots & \dots & p_3 & \dots & \dots \\ -r_{n1} & \dots & \dots & \dots & p_n \end{pmatrix}$$

Elements of the main diagonal in D provide the payables of a trader while the off-diagonal elements in the ith row represent the negatively of the receivables trader i. Off-diagonal elements in the jth column represent the payables of the jth trader to others in the system.

The linear Programming Model

This linear programming model identifies solvent from insolvent traders and determines the debt settlement ratio of all traders.

$$maximize \sum_{i=1}^{n} C_i \lambda_i \tag{1}$$

subject to

$$p_i \lambda_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} r_{ij} \lambda_j \leq a_i, \quad i = 1, 2, ..., n, \tag{2}$$

$$\lambda_i \leq 1, \quad i = 1, ..., n, \tag{3}$$

$$\lambda_i \geq 0, \quad i = 1, ..., n. \tag{4}$$

This model consists of n variables, n constraints, and 2n bounds.

Objective functions is to maximize the weighted sum of DSRs. It is proved that the optimum solution stays the same for any positive value of weights(C_i).

The constraints, (2), forces the solution to satisfy that the amount a trader would actually pay should be less than or equal to the sum of his assets and the amount he actually would receive from other traders.

The 2n bounds, (3) and (4), guarantee that DSRs would be positive, which is a direct result of the definition of DSR, and smaller than one, which means that none of the traders would pay more than he owed.

System of Linear Equations for Debt Payment by Type of Asset

Using matrix notation the asset vector A is given by

$$A = A_1 + A_2 + ... + A_m, where \tag{5}$$

$$A_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ . \\ . \\ . \\ a_{nk} \end{pmatrix},$$

Substituting A in the system of linear equations obtained by setting inequalities (2) to equalities,

$$D\tilde{\lambda} = A_1 + A_2 + ... + A_m$$

Multiplying through by D^{-1} provides the expression:

$$\tilde{\lambda} = D^{-1}A_1 + D^{-1}A_2 + ... + D^{-1}A_m. \tag{6}$$

By setting

$$\tilde{\lambda}_k = D^{-1}A_k, \quad k = 1, ..., m, \tag{7}$$

then

$$\tilde{\lambda} = \tilde{\lambda}_1 + \tilde{\lambda}_2 + ... + \tilde{\lambda}_m. \tag{8}$$

The last equation shows that the debt settlement ratio of a trader equals the sum of the debt settlement ratios for each type of asset owned by this trader. Clearly D^{-1} is independent of the asset type, and we need only a single inversion matrix.

This procedure for debt payment by type of asset mentioned above can also be included in the LP model mentioned above so that it provides directly the vector $\tilde{\lambda}_k$ for each type of asset k, k = 1, 2, ... , m. This is achieved by repeating the set of constraints (2), each containing the vector $\tilde{\lambda}_k$ as variables and the vector A_k as the right-hand side for the asset type k. The payables, p_i, and the receivables, r_{ij}, stay the same regardless of the asset type. Thus, the problem can be expanded to provide $\tilde{\lambda}_k$ as follows:

$$maximize \sum_{i=1}^{N} \sum_{k=1}^{m} C_i \beta_k \lambda_{ik} \tag{9}$$

subject to

$$p_i \lambda_{ik} - \sum_{\substack{j=1 \\ j \neq i}}^{N} r_{ij} \lambda_{jk} \leq a_{ik}, \quad \begin{array}{l} i = 1, ..., N, \\ k = 1, ..., m, \end{array} \tag{10}$$

$$\sum_{k=1}^{m} \lambda_{ik} \leq 1, \quad i = 1, ..., N, \tag{11}$$

$$\lambda_{ik} \geq 0, \quad \begin{array}{l} i = 1, ..., N, \\ k = 1, ..., m. \end{array} \tag{12}$$

This expanded problem has Nm variables and N(m + 1) constraints. Thus this is a much larger problem then the previous one and solving this is also more complex. The resulting m vectors of $\tilde{\lambda}_k$ are the used to compute the actual traders' payables and receivables for each type of asset. This approach ensured a fair and unique method for the disbursement of assets and final settlement of mutual debts among all traders.

3 Discussion

The LP model is a robust quantitative tool to determine DSRs, to identify insolvents, and to measure the DSRs' sensitivity to changes in asset values or the addition of new insolvent traders. The solution obtained was a mathematically exact, simple and effective way to distribute the payments of insolvent traders to their creditors. All creditors were treated equally in that they all received the same DSR from any one single debtor. Unlike the arbitration, the model precisely maximized the sum of the DSRs for all traders so that the whole economy is better off.

In the first stage of the model development, it is assumed that all assets were homogeneous, perfectly substitutable. This was unrealistic, so it was eliminated in the second stage. The traders in the I2 and I3 sets are included in the model, and I1 and I4 traders are excluded for the sake of simplicity and convenience.

The model is transportable to similar post crash situations, where trader payables, receivable, and assets interact.

4 References

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A Linear Programming Model Combining Land Levelling and Transportation Problems

by M. Yuvanç and S. Işılak

1 Verbal Description of the Model

Before establishing any plant (irrigation canals, highways, etc.), engineers have to consider land levelling process which involves a massive movement of earth from one part of the surface to another. Because of such huge replacement of considerable amount of material, engineers must use the right optimization technique to reduce the cost of transportation of levelled earth from the 'cut' site to fill site.

The levelling process is usually done in two stages. The first stage starts by determining the levels of terrain, while the second stage solves the earthwork allocation for the predetermined levels.

In this linear programming model, an alternative technique has been applied to land levelling problem. It's Ignizio's[2] work which is basically a curve fitting technique that this model builds up its approach. In a similar way to curve fitting approach, we are going to use surface fitting technique in which we minimize the code differences to ensure that the final surface is as smooth as possible.

Besides this model is going to take into account the transportation of the material which is to be removed. Therefore, instead of having two problems, namely land levelling and transportation, this very model will combine these two problems to make sure that the global optimum is reached. It's worth emphasizing that the global optimality is not guaranteed by solving the transportation portion of the problem separately as done by Rapsack[3].

Preparation of The Terrain For LP Modeling

Terrain is divided into small rectangles which are either to be levelled (cut) or to be filled. Let the terrain be divided into n rectangles. But it's hard to calculate rectangles in one group for a whole terrain, so we take every group of rectangles which are fitted together as a *plane*.

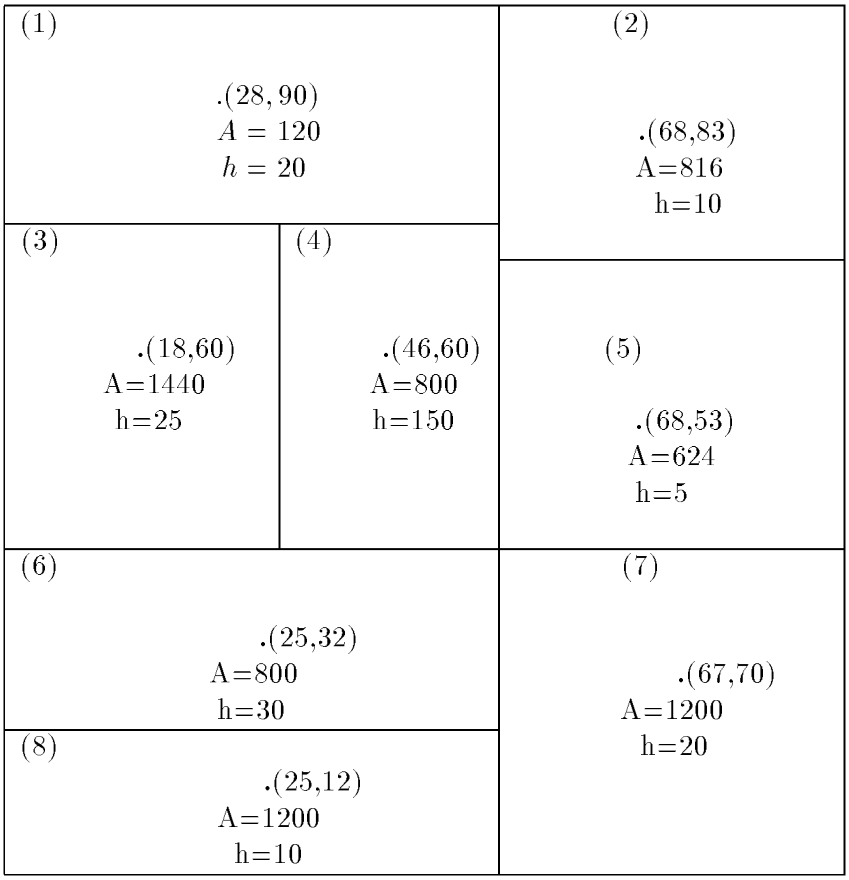

Fig.1: Bold lines separates three planes which are divided into eight rectangles in total.
$A =$ area of rectangle
$h =$ average height of rectangle
$(x, y) =$ coordinates of the center of each rectangle

The model is going to use the height and the rectangular coordinates of each small rectangle as its inputs. Each rectangle is defined as a single point with the average height of the rectangle *(h)* and rectangular coordinates *(x,y)* which are center of the rectangle. In this way, the model assumes all small rectangle as being slopeless.

2 Mathematical Description of the Model

General parameters:

n= number of rectangles in the terrain;

m= number of planes in the terrain;

k= number of touching sides between planes;

q = number of points where more than two planes touch;

a_i= surface area of rectangle *i*;

h_i= the actual average height of rectangle i (corresponding to levelling with zero slope);

(x_i, y_i)= the coordinates of the point in the center of rectangle i .

Decision variables:

u_i = the height of earth to be added to (fill) rectangle *i, (i=1,2,........,n)* ;

v_i = the height of earth to be removed (cut) from rectangle sl i ;

t_{ij}= the number of cubic units of earth to be transported from rectangle *i* to rectangle*j*;

a_i = the slope of plane i in the x-direction ;

b_i = the slope of plane i in the y-direction ;

d_i = the intersection of plane i with the z-axis.

By using a_i, b_i, d_i as decision variables in our model, we can determine their values which would result in the optimum levelling. But this variables can take negative values which we cannot use in the linear modeling . So we modify them as:

$$a_i = a_{i1} - a_{i2}$$

$$b_i = b_{i1} - b_{i2}$$

$$d_i = d_{i1} - d_{i2}$$

Then we get variables which are non-negative.

Cost parameters :

c_{1i} = the cost of adding one cubic unit of earth to rectangle *i*;

c_{2i} = the cost of removing one cubic unit of earth from rectangle *i*;

r_{ij} = the cost of transporting one cubic unit of earth from rectangle i to rectangle j .

Plane parameters :

p_{1i} = the upper bound on the slope of rectangle *i* in the x-direction ;
w_{1i} = the lower bound on the slope of rectangle *i* in the x-direction ;
p_{2i} = the upper bound on the slope of rectangle *i* in the y-direction ;
w_{2i} = the lower bound on the slope of rectangle *i* in the y-direction .

Formulation of The LP Model :

Since we are concerned with the minimization of the cost of land levelling, the objective function has to minimize the filling cost, the cutting cost, and the transportation cost :

$$Minimize\,F \;=\; \sum_{i=1}^{n}(c_{1i}A_iu_i + c_{2i}A_iv_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}t_{ij} \;\;(1)$$

subject to following constraints:

a. First Constraint :

The gap between the original height of rectangle *i* and the height of the *fitted plane* is the result of the filling and cutting applied to *rectangle i* ;

$$a_ix_i + b_iy_i + d_i - h_i \;=\; u_i - v_i \quad i = 1, 2, ..., n \;\;(2)$$

b. Second Constraint :

The total amount of earth added to rectangle *i* minus the total amount of earth levelled from it must be equal to all the earth transported to rectangle *i* from other rectangles minus all the earth transported from it to other rectangles ;

$$\sum_{j=1}^{n}(t_{ji} - t_{ij}) \;=\; A(u_i - v_i) \quad i = 1, 2, ..., n\;;\; i \neq j \;\;(3)$$

c. Third Constraint :

Third constraint is the upper and lower limits of the i^{th} plane's slope ;

$$a_i \;\leq\; p_{1i}$$

$$a_i \;\leq\; w_{1i}$$

$$b_i \;\leq\; p_{2i}$$

$$b_i \;\leq\; w_{2i} \;\;(4)$$

d. Fourth constraint :

The planes have to touch at least two points since we need to have a smooth transition from one plane to the adjacent one ;

$$a_i x^1 + b_i y^1 + d_i \ = \ a_j x^1 + b_j y^1 + d_j$$

$$a_i x^2 + b_i y^2 + d_i \ = \ a_j x^2 + b_j y^2 + d_j \ \ (5)$$

Dimensions of the basis :

Now that we put the model into standard LP format, we can easily calculate the number of columns and the rows of the basis :

(1) Number of columns (variables) is equal to $n^2 + n + 6m$:

(a) From equation (2), u_i and v_i account for *2n* columns;

(b) From equation (3), the transportation variables t_{ij} account for *n(n-1)* columns;

(c) From equation (4), the variables *a, b, d* account for *6m* columns.

(2) Number of rows (constraints) is equal to $2n + 4m + 2kq$:

(a)From equation (2), the plane constraints account for *n* rows;

(b)From equation (3), the transportation constraints account for *n* rows;

(c)From equation (4), the limit on slopes account for *4n* rows;

(d)From equation (5), the touching of the plane edges account for $2k - q$ rows.

3 Discussion

Let's compare this model with Rapsack's model: Rapsack studied with the same problem and reported his application which involves 9 planes and 295 rectangles. He solved this problem with variables and constraints which took an amount of 1359 rows 1949 columns.

If we apply the same problem to our model, we see that our total number of variables is

$$2n + 6m \ = \ 2 \times 295 + 6 \times 9 \ = \ 644$$

columns (variables). The number of rows (constraints) is $n + 4m + (2k - q)$, where the value of the term $(2k - q)$ is dependent on the arrangements of planes with respect to each other, thus amounting to a maximum number of nodes

$$295 + 4 \times 9 + 2 \times 12 \ = \ 355$$

The reduction in problem size is obvious: a 65% having the number of columns and more than 60% in the number of the rows. However, we only compared the levelling stage. Our model also considers transportation of the earthwork. When we take transportation Part into account, the number of columns (variables) amount to

$$n^2 + n + 6m \ = \ 87374,$$

with the maximum number of rows (constraints) being

$$2n + 4m + (2k - q) \ = \ 2 \times 295 + 4 \times 9 + 24 = 650$$

.

It should be worth noting here that if we solve the transportation part separately from the linear program (as done by the Rapsack), we will have to deal with 86,730 variables and 590 constraints.

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