1. [1 point] Prove or disprove the following statement:

If \( V \) is a real inner product space, then for arbitrary non-zero vectors \( v \) and \( w \) in \( V \), we have

\[
-1 \leq \frac{\langle u, v \rangle}{\|u\|\|v\|} \leq 1.
\]
2. [3 points] Find the closest vector to \((2, 1, 2, -1, -2)\) in the subspace spanned by \((1, 1, 0, 0, -1), (1, -1, 1, 0, 0), (0, 0, 0, 1, 0)\).
3. [3 points]
Prove or disprove the following statement:

If \( \{v_1, \ldots, v_n\} \) is an orthogonal set of vectors of an inner product space \( V \), then for each \( v \in V \) we have

\[
\|v\| \geq \sqrt{\sum_{k=1}^{n} \frac{(v, v_k)^2}{\langle v_k, v_k \rangle}}.
\]

Equality holds iff

\[
v = \sum_{k=1}^{n} \frac{(v, v_k)^2}{\langle v_k, v_k \rangle} v_k.
\]
4. [2 points] Find the orthogonal projection of the function $e^x$ on the subspace of $C[-\pi, \pi]$ (with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g}(x)dx$) generated by $\cos x$ and $\sin x$. (Recall that $\sinh x = \frac{e^x - e^{-x}}{2}$).
5. [4 points] We are given the vector space of continuous functions on $[-1, 1]$, the space $C[-1, 1]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\bar{g}(x)dx$. Let $V_+ = \{ f \in C[-1, 1] : f(x) = f(-x) \}$ be the subspace of even functions and $V_- = \{ f \in C[-1, 1] : f(x) = -f(-x) \}$ be the subspace of odd functions. Prove or disprove:

$$V = V_+ \oplus V_-.$$
6.[4 points] Consider a complex vector space $V$ of dimension $n$. You are given two operators $F$ and $G$ on $V$ such that

$$GF - FG = \alpha F$$

where $\alpha$ is a complex nonzero number. Let $v$ be an eigenvector of $G$ corresponding to eigenvalue $\lambda$.

(a) Show (using induction argument) that for all integer $k$ ($k = 1, \ldots$) we have

$$GF^k(v) = (\lambda + \alpha k)F^k(v)$$

(b) Deduce from (a) the existence of an integer $k \in \{0, 1, \ldots, n\}$ such that $F^k(v) = 0$ and that $F$ is not injective.
7. [3 points] We are given \( n \) real nonzero numbers \( b_1, \ldots, b_n \) and the \( n \times n \) matrix \( A \) where the \((i, j)\) entry \( a_{ij} \) is given by

\[
a_{ij} = b_i b_j.
\]

Therefore, \( A \) is the matrix representation of an operator \( T \) of a real \( n \)-dimensional vector space \( V \) in a given basis \( \{e_1, \ldots, e_n\} \).

Using the definition of an eigenvalue and eigenvector, show that \( \lambda_1 = 0 \) is an eigenvalue of \( T \) and that the eigenspace \( E_1 \) associated with \( \lambda_1 \) is a subspace of dimension \( n - 1 \) that you should determine explicitly.