The Robust Merton Problem of an Ambiguity Averse Investor

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Abstract

In a continuous-time financial market composed of \( N \) risky assets and a riskless asset, where short sales are allowed and utility maximizing investors can be ambiguity averse, i.e., diffident about mean return estimates where confidence is represented using ellipsoidal uncertainty sets, we derive a closed form portfolio rule based on a worst case max-min criterion which extends the classical Merton problem in continuous time, and reverts to the classical Merton solution for an ambiguity-neutral investor. The portfolio allocation policy of a reasonably ambiguity-averse investor is shaped by a modified market Sharpe ratio.

**Keywords**: Robust optimization, Merton problem, ellipsoidal uncertainty, Hamilton-Jacobi-Bellmann equation.

**AMS subject classifications**: 91G10, 91B25, 90C25, 90C46, 90C47

1 Introduction

The purpose of the present note is to extend the technique of worst-case (max-min) robust portfolio choice (see e.g., [1, 2] for seminal papers on robust optimization, and [6, 7, 8, 9, 10, 11, 16, 17] for robust portfolio optimization in single period problems) for an ambiguity-averse investor under ellipsoidal representation of ambiguous parameters (in the continuous-time setting, the parameter of concern is usually the volatility whereas here the focus will be the growth rate, following the footsteps of static robust portfolio optimization literature e.g., [10]) into a continuous-time portfolio optimization setting commonly known as the Merton problem [14, 15, 20]. The variant of the Merton problem treated here will be referred to as the robust Merton problem under ambiguity aversion. There is a growing literature on decision making under ambiguity aversion, see e.g., [12, 13] and the references cited therein. A related line of literature is concerned with Knightian uncertainty (in the sense that the distribution of

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1The sensitivity of portfolio holdings to imprecision in mean return is well documented, see e.g., [3, 4].
uncertain parameters such as volatility is not known) and specification of ambiguity by multiple priors; see [18] for a recent work as well as the many references therein. While the solution to the robust Merton problem that we shall offer is simple and entirely based on a recent (text book) exposition of L.C.G. Rogers [19] we believe in the value of circulating this problem and its solution as we remain unaware of a similar exercise in the literature, to the best of our knowledge. There is one exception to this statement, though. In section 2.33 of [19], the growth rate for the risky asset in a two asset model (one risky and one riskless) is assumed to be unknown but to lie in an interval of uncertainty. In this case, the robust policy according to [19] is a somewhat uninteresting policy, i.e., it is to keep all the wealth in the risky asset while, as will be shown below, in the case of ellipsoidal uncertainty, we obtain a non-trivial robust (worst-case) investment policy for a reasonably ambiguity-averse investor (quantified by relative magnitudes of a certain coefficient of ambiguity and the optimal market Sharpe ratio), which reduces to the classical Merton solution when the ambiguity in the growth rate vector is not an issue, i.e., when the investor is ambiguity-neutral. It is hoped that this paper will fuel further research on continuous-time robust portfolio optimization.

Organization of this note is as follows. In the next section, the problem is laid out with some generalities. In section 3 the solution is derived for the infinite horizon version. In section 4, the finite horizon version is briefly discussed.

2 The Robust Merton Problem under Ambiguity Aversion

Consider the problem of an agent investing in \( N \) risky assets and a riskless asset, and his wealth \( w_t \) process described by the following equation

\[
\begin{align*}
\text{dw}_t &= r_t w_t \, dt + n_t \cdot (dS_t - r_t S_t + \delta_t \, dt) + e_t \, dt - c_t \, dt \\
&= r_t (w_t - n_t \cdot S_t) \, dt + n_t \cdot (dS_t + \delta_t \, dt) + e_t \, dt - c_t \, dt
\end{align*}
\]

where the asset price process \( S_t \) is a \( N \)-dimensional semi-martingale, the portfolio process \( n \) is a \( N \)-dimensional predictable process and the dividend process \( \delta_t \) is a \( N \)-dimensional adapted process. The adapted scalar processes \( e \) and \( c \) are respectively an endowment stream and consumption stream. In what follows we shall assume the endowment stream and dividend process to be identically zero. Here, the components of \( n_t \) represent the number of shares of assets held in the portfolio. Passing to the representation\(^2\) \( \theta = n \cdot S \), we have the worth (cash value) of each asset in the portfolio stored in the components of the \( N \)-dimensional process \( \theta \).

\(^2\) We use \( x \cdot y \) and \( x^T y \) interchangeably to denote the inner product between two vectors \( x \) and \( y \) of conformable dimensions; we also use \( A \cdot b \) to denote the matrix vector product of matrix \( A \) with vector \( b \). The meaning should be sufficiently clear from context.
Specifically, assume that the asset dynamics are as follows for each asset \( we = 1, \ldots, N \):

\[
dS^i_t = S^i_t \left( \sum_{j=1}^{d} \sigma^{ij} dW^j_t + \mu^i dt \right)
\]

(3)

where \( \sigma^{ij} \) and \( \mu^i \) are constants and \( W \) is a \( d \)-dimensional Brownian motion. The riskless rate \( r \) is assumed constant. In matrix-vector form the above equation is expressed as

\[
dS_t = S_t (\sigma \cdot dW + \mu dt)
\]

(4)

where \( \sigma \) is an \( N \times d \) matrix, and \( \mu \) is a \( N \)-vector. Now, the equation governing the evolution of wealth can be recast as:

\[
dw_t = rw_t dt + \theta_t \cdot (\sigma \cdot dW_t + (\mu - r1) dt) - c_t dt,
\]

(5)

where \( 1 \) denotes a vector with all components equal to one. We shall say that the pair \( (\theta_t, c_t) \) is admissible for the initial wealth \( w_0 \) if the wealth process \( w_t \) given by (5) remains non-negative at all times. Let \( \mathcal{A}(w_0) \) be the set of all admissible \( (\theta, c) \) pairs for initial wealth \( w_0 \). The agent is trying to choose \( (\theta, c) \in \mathcal{A}(w_0) \) so as to maximize

\[
\mathbb{E} \left[ \int_0^T u(t, c_t) dt + u(T, w_T) \right]
\]

where the utility function \( u \) is concave increasing in the second argument and measurable in the first.

The exposition so far has followed the opening chapter of Rogers [19] which gives in section 1.2 a solution approach of a remarkable simplicity. We shall adopt his approach in solving the problem posed in the present paper. First, we state the problem. We shall assume that the agent is diffident about the mean return estimates \( \mu^i \) of risky assets. It is usually true that the mean return estimates are subject to imprecision due to inaccuracy of forecasts, lack of sufficient data etc. The investor is ambiguity averse with ambiguity aversion coefficient \( \epsilon \) such that his/her confidence in the mean return vector estimate is expressed as a belief that the true mean return lies in the ellipsoidal set

\[
U_\mu = \{ \mu \| (\sigma \sigma^T)^{-1/2} (\mu - \hat{\mu}) \|_2 \leq \epsilon \},
\]

that is, an \( N \)-dimensional ellipsoid centered at \( \hat{\mu} \) (the estimated mean return vector) with radius \( \epsilon \). The idea is that the decisions of an ambiguity averse investor are made by considering the worst case occurrences of the true mean return \( \mu \) within the set \( U_\mu \). Therefore, more conservative portfolio choices are made when the volume of the ellipsoid is larger, i.e. for greater values of \( \epsilon \), while an ambiguity-neutral investor with no doubt about errors in the estimated values sets \( \epsilon \) equal to zero. The merits of an ellipsoidal representation for the ambiguity set has been amply demonstrated and discussed in [2, 10, 11]. Hence, we do not repeat them here. Intuitively, the non-linear but simple geometry of ellipsoids offers robustness that avoids a worst case where all
ambiguous elements assume their worst behaviour simultaneously (which would be the case, e.g., in a polyhedral hyper-rectangle or box representation) while it preserves tractability. We cite Fabozzi et al. [7] for a convincing argument for its adoption: “The coefficient realizations are assumed to be close to the forecasts, but they may deviate. They are more likely to deviate from their means if their variability (measured by their standard deviation) is higher, so deviations from the mean are scaled by the inverse of the covariance matrix of the uncertain coefficients. The parameter $\epsilon$ corresponds to the overall amount of scaled deviations of the realized returns from the forecasts against which the investor would like to be protected.”

Thus, the ambiguity-averse investor sets out to solve the following (finite or infinite-horizon if $T = \infty$) robust Merton problem under ambiguity of mean returns $\mu$:

$$\sup_{(\theta, c)} \mathbb{E}\left[\int_0^T u(t, c_t)dt + u(T, w_T)\right]$$

over

$$(\theta, c) \in A(w_0) \forall \mu \in U_{\mu},$$

where the wealth process evolves according to

$$dw_t = rw_t dt + \theta_t \cdot (\sigma \cdot dW_t + (\mu - r1) dt) - c_t dt.$$

3 The Solution for Power Utility: Infinite Horizon

We shall say that the pair $(\theta, c_t)$ is robust admissible for initial wealth $w_0$ if the wealth process remains non-negative for all $\mu \in U_{\mu}$ at all times. We shall write $A^U(w_0)$ as the set of robust admissible pairs $(\theta, c)$. We shall base the solution on the following result known as the Davis-Varaiya Martingale Principle of Optimal control; see [19].

**Theorem 1** Suppose that there exists a function $V : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ which is $C^{1,2}$, such that $V(T,.) = u(T,.)$. Suppose also that for any $(\theta, c) \in A^U(w_0)$

$$Y_t \equiv V(t, w_y) + \int_0^t u(s, c_s)ds$$

is a supermartingale, and that for some $(\theta^*, c^*) \in A^U(w_0)$ the process $Y$ is a martingale. Then $(\theta^*, c^*)$ is optimal for the problem

$$\sup_{(\theta, c) \in A^U(w_0)} \mathbb{E}\left[\int_0^T u(t, c_t)dt + u(T, w_T)\right]$$

where the wealth process evolves according to

$$dw_t = rw_t dt + \theta_t \cdot (\sigma \cdot dW_t + (\mu - r1) dt) - c_t dt,$$
and the value of the problem starting from the initial wealth \( w_0 \) is

\[
V(0, w_0) = \sup_{(\theta, c) \in \mathbb{A}^U(w_0)} \mathbb{E}\left[ \int_0^T u(t, c_t) dt + u(T, w_T) \right].
\]

Now, we shall use the above theorem to solve the ambiguity-averse investor’s problem in the infinite horizon case \( (T = \infty) \) first, as follows. We look for a function \( V \) satisfying the premises of the theorem. Using Itô expansion, we have

\[
dY_t = V_t \cdot \sigma dW + \left\{ u(t, c_t) + V_t + V_w(rw + \theta \cdot (\mu - r1) - c) + \frac{1}{2} \|\sigma^T \theta\|_2^2 V_{ww} \right\} dt
\]

for any \( \mu \). To make the process \( Y \) a supermartingale we shall compute the supremum over \( \theta, c \) of the infimum of the drift term above over all \( \mu \in U_\mu \) and equate it to zero. i.e., we shall deal with the max-min equation

\[
0 = \sup_{\theta, c} \inf_{\mu \in U_\mu} \left\{ u(t, c_t) + V_t + V_w(rw + \theta \cdot (\mu - r1) - c) + \frac{1}{2} \|\sigma^T \theta\|_2^2 V_{ww} \right\}.
\]

Note that this max-min HJB equation is also the approach taken in [18] to deal with ambiguity associated with multiple priors. We solve the inner inf problem in closed form first. The solution of the convex optimization problem

\[
\min_{\mu} \{ \theta^T \mu : (\mu - \hat{\mu})^T (\sigma \sigma^T)^{-1} (\mu - \hat{\mu}) \leq \epsilon^2 \}
\]

is a simple exercise in Karush-Kuhn-Tucker necessary and sufficient conditions, and is obtained as

\[
\mu^* = \hat{\mu} - \frac{\sigma \sigma^T \theta}{\|\sigma \sigma^T\|_2^{1/2}},
\]

which, substituted back, gives:

\[
0 = \sup_{\theta, c} \left\{ u(t, c_t) + V_t + V_w(rw + \theta \cdot (\hat{\mu} - r1) - \epsilon \sqrt{\theta^T \sigma \sigma^T \theta} - c) + \frac{1}{2} \|\sigma^T \theta\|_2^2 V_{ww} \right\}.
\]

(7)

Notice that the above HJB equation (7) is equivalently viewed as stemming from the worst-case \( \mu \) in \( U_\mu \), namely \( \mu^* \) given above, and using it in the wealth equation

\[
dw_t = rw_t dt + \theta_t \cdot \sigma \cdot dW_t + [\theta_t \cdot (\hat{\mu} - r1) - \epsilon \sqrt{\theta_t^T \sigma \sigma^T \theta_t}] dt - c_t dt.
\]

(8)

Now, we assume the following form for the utility function (CRRA power utility)

\[
u(t, x) = e^{-\rho t} \frac{x^{1-R}}{1-R}
\]

as in [19], where \( \rho \) and \( R \) are some positive constants. In the infinite horizon case we wish to find the value function \( V \) defined as:

\[
V(w) = \sup_{\theta, c \in \mathbb{A}^U(w_0)} \mathbb{E}\left[ \int_0^\infty e^{-\rho s} \frac{x_s^{1-R}}{1-R} ds \right],
\]

5
(\( V \) is concave increasing by virtue of concavity and increasing nature of \( u \)). Using the specific utility function we arrive at the guess

\[
V(w) = \gamma e^{-R u(w)} = \gamma e^{-R w^{1-R}} \frac{1}{1-R}
\]

for some constant \( \gamma \) to be determined (we use \( \epsilon \) as subscript to highlight the dependence on the radius of ambiguity \( \epsilon \)). The above form is obtained as follows. Consider \( V(t, w) \) which is defined as

\[
V(t, w) = \sup_{\theta, c \in \mathcal{A}^\epsilon(w_0)} \mathbb{E} \left[ \int_t^\infty e^{-\rho s} \frac{c^{1-R}}{1-R} ds | w_t = w \right].
\]

The time-homogeneity of the problem implies

\[
V(t, w) = -e^{-\rho t} V(w),
\]

and using the scaling property (see Proposition 1.2 pp. 7 of [19]) the guess at the value function takes the form

\[
V(t, w) = e^{-\rho t} \gamma e^{-R u(w)}.
\]

With this guess we return to (7) to do the optimization over \( \theta \) and \( c \). The optimization over \( c \) is carried out exactly using convex conjugate duality as in [19] (pp. 8–9) and results in

\[
c^*_{\epsilon} = \gamma_\epsilon w,
\]

with

\[
\sup_c \{ u(t, c) - c V_w \} = e^{-\rho t} R \frac{1-R}{1-R} (\gamma_\epsilon w)^{1-R}.
\]

On the other hand the optimization over \( \theta \) yields:

\[
\theta^*_{\epsilon} = -\frac{V_w(H - \epsilon)}{V_{ww}H} (\sigma \sigma^T)^{-1}(\hat{\mu} - r1).
\]

where \( H = \sqrt{(\hat{\mu} - r1)^T (\sigma \sigma^T)^{-1}(\hat{\mu} - r1)} \) is the maximum Sharpe ratio of the market (\( \kappa = (\sigma \sigma^T)^{-1/2}(\hat{\mu} - r1) \) is referred to as the market price of risk\(^3\) vector in the literature\(^4\), with \( \hat{\mu} \) replaced by the given \( \mu \)). We define the ambiguity-adjusted Sharpe ratio \( H_\epsilon \) as \( H - \epsilon \). To see the above portfolio result, consider the first-order conditions (the function to be maximized is concave, so the first-order conditions are necessary and sufficient) which yield

\[
\theta^*_{\epsilon} = -\frac{s V_w}{s V_{ww} - V_w \epsilon} (\sigma \sigma^T)^{-1}(\hat{\mu} - r1)
\]

\(^3\)In [19] the definition of \( \kappa \) is \( \sigma^{-1}(\hat{\mu} - r1) \), which does not make sense unless \( N = d \).

\(^4\)A standard result from Mean-Variance portfolio theory shows that the slope of the Capital Market Line, given by \( H \), is equal to the maximal Sharpe ratio, i.e.,

\[
H = \max_\omega \frac{\hat{\mu}^T \omega}{\sqrt{\sigma^2 \sigma^T \omega}}
\]

where \( \hat{\mu} = \hat{\mu} - r1 \).
where \( s = \sqrt{\theta^T (\sigma \sigma^T) \theta} \). Solving for \( s \) from the previous defining equation yields the positive root
\[
s^* = \frac{V_w (H - \epsilon)}{V_{ww}}
\]
which is positive provided that \( H > \epsilon \) (recall that by virtue of concavity \( V_{ww} < 0 \) and by the increasing property of \( V \), \( V_w > 0 \)) which we shall assume henceforth to be the case. An investor with an ambiguity aversion \( \epsilon \) less than the market Sharpe ratio \( H \) is termed a \textit{reasonably ambiguity-averse} investor. Substituting \( s^* \) back we obtain the announced solution \( \theta^*_\epsilon \).

Using the suspected form of \( V \) we have
\[
\theta^*_\epsilon = \frac{w H}{R H} (\sigma \sigma^T)^{-1} (\hat{\mu} - r 1).
\]
Notice that the optimal portfolio \( \theta^*_\epsilon \) preserves the essential form of the Mutual Fund theorem of Merton in that the optimal portfolio consists of an allocation between two fixed mutual funds, namely the riskless asset and the fund of risky assets given by \((\sigma \sigma^T)^{-1} (\hat{\mu} - r 1)\). At each time point the optimal relative allocation of wealth is now dependent on the ambiguity aversion of the investor in addition to his/her risk aversion through the coefficient:
\[
\frac{w H}{R H}.
\]
The above allocation naturally collapses to the Merton allocation for \( \epsilon = 0 \). In case the radius of ambiguity \( \epsilon \) is greater than or equal to the market Sharpe ratio \( H \), the optimal control policy is not to invest at all into the risky assets.

Define
\[
\pi^\epsilon = \frac{H}{R H} (\sigma \sigma^T)^{-1} (\hat{\mu} - r 1).
\]
Hence, the robust Merton portfolio of a reasonably ambiguity-averse agent has the form
\[
(\theta^*_\epsilon)_i = w_i \pi^\epsilon_i, \forall i = 1, \ldots, N.
\]
Note that ambiguity aversion results in smaller portfolio positions in absolute value with respect to the classical Merton portfolio with \( \epsilon = 0 \), i.e., both long and short positions are shrunk with respect to the ambiguity-neutral portfolio. Similarly for consumption, since \( \gamma^\epsilon \) is smaller than \( \gamma_M \) of Merton the consumption in the ambiguity averse case is also curtailed even for identical wealth levels.

Now, we substitute these \( c^*_\epsilon \) and \( \theta^*_\epsilon \) back into the equation (7) and solve for the constant \( \gamma^\epsilon \). Straightforward calculations result in:
\[
\gamma^\epsilon = \frac{\rho + (R - 1) (r + \frac{\epsilon H}{R} + \frac{1}{2} \frac{H^2}{R})}{R}.
\]
Clearly for \( \epsilon = 0 \) we fall back to the constant \( \gamma_M \) of [19], (see pp. 9 [19], eqn. (1.29)). Therefore, the value function \( V \) of the problem is found as
\[
V(t, w) = \gamma^\epsilon R u(w).
\]
Assuming that $\gamma_\epsilon$ is positive (it suffices that $R$ is larger than one for this property to hold; in fact as Proposition 1.3 of [19] shows, the Merton problem is ill-posed if $\gamma_M < 0$), we evaluate the evolution of the wealth process $w$ for the worst-case choice of $\mu$, c.f. eqn. (8):

$$dw_t = rw_t dt + \theta_t \cdot \sigma \cdot dW_t + [\theta_t \cdot (\hat{\mu} - r1) - \epsilon \sqrt{\theta_t^T \sigma \sigma^T \theta_t}] dt - c_t dt.$$

Substituting the candidate point $(\theta^*_\epsilon, c^*_\epsilon)$ into the above and using straightforward algebra we obtain

$$dw_t = w^*_t \left\{ \left[ r + \frac{H^2}{R} - \gamma_\epsilon \right] dt + \frac{H_t}{HR^{\kappa}} \cdot dW_t \right\},$$

which results in the solution

$$w^*_t = w_0 e^{\frac{H^2}{R} \cdot W_t + [r + \frac{H^2 (2R-1)}{2R^2} - \gamma_\epsilon]},$$

The above wealth process reverts to the optimal wealth process of an ambiguity-neutral agent as in [19] for $\epsilon = 0$.

Optimality of the solution above can be verified exactly as done on pp. 10-11 of [19]. This verification, based on obtaining a simple bound on the utility function using the gradient inequality for a concave function and then showing that the bound is attained at the candidate solution, is almost verbatim repetition of steps 3 and 4 on pages 10-11 of [19], so is omitted.

**Proposition 1** The infinite-horizon robust Merton problem under ambiguity of mean returns $\mu$ for a reasonably ambiguity-averse agent ($\epsilon < H$):

$$\sup_{(\theta, c)} \mathbb{E} [\int_0^\infty u(t, c_t) dt]$$

(with CRRA power utility) over

$$(\theta, c) \in \mathcal{A}(w_0) \forall \mu \in U_\mu,$$

where the wealth process evolves according to

$$dw_t = rw_t dt + \theta_t \cdot (\sigma \cdot dW_t + (\mu - r1) dt) - c_t dt$$

admits the optimal controls:

$$\theta^*_{\epsilon,t} = \frac{w^*_t H_t}{RH} (\sigma \sigma^T)^{-1} (\hat{\mu} - r1)$$

$$c^*_{\epsilon,t} = \gamma_\epsilon w^*_t,$$

with

$$\gamma_\epsilon = \frac{\rho + (R-1)[r + \frac{\epsilon H_t}{R} + \frac{1}{2} \frac{H^2_t}{R}]}{R},$$

$$w^*_t = w_0 e^{\frac{H^2}{R \cdot W_t} + [r + \frac{H^2 (2R-1)}{2R^2} - \gamma_\epsilon]}.$$
4 The Finite Horizon Problem

Consider now the problem of a reasonably ambiguity-averse investor (i.e., $H > \epsilon$) in a finite horizon environment as in Section 2.1 [19]:

$$\sup_{\theta, c \in A^U(w)} \mathbb{E}\left[ \int_0^T h(t)u(c_t)dt + Au(w_T) \right]$$

for some strictly positive function $h$ (e.g., $e^{-\rho t}$) and constant $A > 0$ with the CRRA utility $u(x) = \frac{x^{1-R}}{1-R}$ for some positive $R \neq 1$. Using the scaling properties of the CRRA utility the value function

$$V(t, w) = \sup_{\theta, c \in A^U(w)} \mathbb{E}\left[ \int_0^T h(t)u(c_t)dt + Au(w_T) \right]_{w_t = w}$$

has the form $V(t, w) = f(t)u(w)$. Following the development of [19] with the necessary changes we solve the HJB equation:

$$0 = \sup_{\theta, c} [u(t, c) + V_t + V_w(rw + \theta \cdot (\hat{\mu} - r1) - \epsilon \sqrt{\theta^T \sigma \sigma^T \theta} - c) + \frac{1}{2} \|\sigma \theta\|^2_{V_{ww}}],$$

as in the previous section to obtain:

$$\theta_\epsilon^* = \pi_\epsilon w_t, \quad c_\epsilon^* = w_t \left( \frac{h(t)}{f(t)} \right)^{1/R}.$$

Substituting the above into the HJB equation after some straightforward calculations we obtain an ODE that is solved as on pp. 31 of [19] to give

$$f(t) = g(t)^R$$

with

$$g(t) = e^{b_\epsilon t} \left[ e^{-b_\epsilon t} A^{1/R} + \int_t^T e^{-b_\epsilon s} h(s)^{1/R} ds \right],$$

where

$$b_\epsilon = \frac{(R - 1)}{R} \left( r + \frac{H^2}{2R} \right).$$

Comparing this to the solution of [19] the only changes are in the constant $b_\epsilon$ where $H^2$ (or $\kappa^2$ in the terminology of [19]) is replaced by $H^2_\epsilon$ and the optimal portfolio allocation which is identical to the robust allocation case of the previous section. Obviously for an ambiguity neutral investor with $\epsilon = 0$ we fall back to finite horizon solution of Merton; c.f. [19] section 2.1, pp. 30–31.

**Proposition 2**

The finite-horizon robust Merton problem under ambiguity of mean returns $\mu$ for a reasonably ambiguity-averse agent ($\epsilon < H$):

$$\sup_{(\theta, c)} \mathbb{E}\left[ \int_0^T h(t)u(c_t)dt + Au(w_T) \right]$$
(with \( h \) a positive function, and positive constants \( A \) and \( R \not= 1 \), and a CRRA power utility), over
\[
(\theta, c) \in \mathcal{A}(w_0) \forall \mu \in U_\mu,
\]
where the wealth process evolves according to
\[
dw_t = rw_t dt + \theta_t \cdot (\sigma \cdot dW_t + (\mu - \alpha) dt) - c_t dt
\]

admits the optimal controls:
\[
\theta^*_t = \frac{w^*_t H_t}{R H} (\sigma \sigma^T)^{-1} (\hat{\mu} - \alpha)
\]
\[
c^*_t = w^*_t \Big( \frac{h(t)}{f(t)} \Big)^1/R
\]

with
\[
f(t) = g(t)^R
\]

where
\[
g(t) = e^{b_t t} \left[ e^{-b_t t} A^{1/R} + \int_t^T e^{-b_s s} h(s)^{1/R} ds \right],
\]

and
\[
b_t = \frac{(R - 1)}{R} \left( r + \frac{H^2}{2R} \right).
\]

5 Concluding Remarks

In this brief paper, we developed a robust max-min version of Merton problem under ambiguity aversion that is understood as aversion to imprecision in the asset growth rates represented as members of an ellipsoidal set. The ambiguity aversion level is controlled by a parameter that acts on the volume of the set. Using a recent monograph exposition for the solution of the classical Merton problem, solutions to the robust problem in both infinite horizon and finite horizon versions under a CRRA utility function were derived. The solution reveals that the investment behaviour under reasonable ambiguity aversion is influenced by a modified market Sharpe ratio in that the ambiguity aversion tends to diminish the market Sharpe ratio viewed by the investor. The robust policy reduces to the classical Merton policy for an ambiguity-neutral investor.

References


