Mechanism Design: An Introduction from an Optimization Perspective

Mustafa Ç. PINAR

Bilkent University

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Basics of Mechanism Design
- Generalities

Selling a Single Indivisible Good
- The Direct Mechanism
- The Revelation Principle
- Numerical Examples

Basics of Shortest Paths
- Formulation

Optimal Mechanism via Shortest Paths

Robust Direct Mechanisms
- Reformulation
- A Stability Result

Acknowledgement

What is Mechanism Design?

- It is concerned with optimizing the allocation of resources
- in situations where one of the parties (or several parties) hold private information that influences the outcome of the game or negotiation..
- Mechanism design allows the planner to simultaneously elicit private information and choose the optimal allocation.

Examples

- Many mechanism design problems are optimization problems
- Auctions (e.g., spectrum allocation)
- Matching (e.g., elective course allocation to students)
- How to divide an indivisible asset? (when the 500 years old Université catholique de Louvain split into two universities in the 1970s, flemish speaking and french speaking, what happened to the library?
- What happened to the national library when India and Pakistan split?
- Answer: unfortunately King Solomon's justice was implemented (The Old Testament)

King Solomon's Justice: The Old Testament

Then spake the woman whose the living child was unto the king, for her bowels yearned upon her son, and she said, O my lord, give her the living child, and in no wise slay it.
1 Kings 4, 26

Bertolt Brecht plagiarized the Old Testament

Ihr aber, ihr Zuhörer der Geschichte vom Kreidekreis
Nehmt zur Kenntnis die Meinung der Alten:
Daß da gehören soll, was da ist, denen, die für es gut sind, also
Die Kinder den Mütterlichen, damit sie gedeihen
Die Wagen den guten Fahrern, damit gut gefahren wird
Und das Tal den Bewässerern, damit es Frucht bringt.

Bertolt Brecht, Der kaukasische Kreidekreis, 1945

Our Setting

- Pricing a single indivisible good
- A seller seeks to sell a single indivisible good
- The seller herself does not attach any value to the good
- Her objective is to maximize the expected revenue from selling the good
- She is thus risk neutral

The direct mechanism:

- The seller wants to adopt a selling mechanism to maximize expected profits
- Parameters (called *type*) needed to determine an optimal allocation are privately held by agents who will consume the resources to be allocated
- Those parameters determine the utility an agent will enjoy from a particular allocation
- In a direct mechanism agents report a type
- The reported type influences the allocation and the utility of the agent

The direct mechanism II:

- Seller will decide two functions with argument t (decision variables of the mechanism):
- the allocation rule $\mathcal{A}_t$: the fraction of the object that goes to the buyer or better: the probability that the buyer will get the good
- the payment rule: p_t (the payment of the buyer to the seller).

The buyer

- There is just one potential buyer (the agent)
- Value that a buyer assigns to the good is called his *type*, denoted t
- The buyer is risk neutral with respect to money
- The buyer's utility if he gets the good and pays a monetary transfer p to the seller is $t - p$
- His utility if he does not get the good is zero.

Crucial assumption

- The value t is known to the buyer but it is not known to the seller!
- Types are assumed to be independent draws from $T = \{1, \dots, m\}$
- The probability mass: $f_t > 0$ is probability that buyer is of type t
- The cumulative distribution function: $F(t) = \sum_{s \leq t} f_s$
- The (inverse) hazard function: $\frac{1-F(t)}{f_t}$
- The *virtual value* $\nu(t) = t - \frac{1-F(t)}{f_t}$ of type t (Myerson 1981)
- The monotonicity of $\nu(t)$ will be important! More on this issue later..

Why does it work?

- Legitimate question: Could there be other mechanisms?
- Certainly! The seller could negotiate with the agent
- The seller could offer the agent a lottery
- or imagine any other convoluted procedure..

Revelation Principle

- In fact, there is no need for any of these complicated mechanisms
- There is a famous result known as the "Revelation Principle"
- It allows to simplify the analysis because
- it shows WLOG that the search for the optimal mechanism can be restricted to direct mechanisms, that is search for pairs of functions p and $\mathcal{A}$
- where the buyer finds it optimal to truthfully report her type.

How does it work?

- The buyer should be compelled to reveal her type truthfully (*Incentive compatibility*)
- Buyer should have an incentive to participate (*Individual rationality*)
- Seller should maximize expected revenue
- These are the constraints and objective function of our optimization problem!

The formulation

- The seller (planner) will choose the optimal mechanism according to the optimal solution of the following LP problem:

$$
\begin{aligned}
\max \quad & \sum_{t=1}^{m} f_t p_t \\
\text{s.t.} \quad & t\mathcal{A}_t - p_t \geq t\mathcal{A}_s - p_s, \forall\, t, s \in T \ (IC) \\
& t\mathcal{A}_t - p_t \geq 0,\ \forall t \in T \ (IR) \\
& \mathcal{A}_t \in [0, 1]\ \forall t \in T
\end{aligned}
$$

- The first constraint is Incentive Compatibility; the second is Individual Rationality; we maximize expected revenue.
- The allocation is continuous but will take discrete values (we shall see this later).

A Numerical Example

- Consider the sale of a single good
- where buyers can be of five types $\{1, 2, \ldots, 5\}$.
- the prob. mass function $f = (0.1, 0.2, 0.2, 0.3, 0.2)$
- verify that $\nu(t)$ is monotone in t
- it has values $(-8, -3/2, 1/2, 10/3, 5)$
- set up and solve the LP
- the optimal solution is $\mathcal{A}_1 = \mathcal{A}_2 = 0$
- $\mathcal{A}_3 = \mathcal{A}_4 = \mathcal{A}_5 = 1$
- $p_3 = p_4 = p_5 = 3$, $p_1 = p_2 = 0$

A Numerical Example

- What does the solution say?
- the seller should set the price equal to 3
- any buyer who reveals a type of 3, 4 or 5 can pay the price and take home the good!
- but what do you notice?
- the optimal price occurs at the smallest type t at which a change of sign occurs in the virtual value $\nu(t)$.
- This is not a coincidence!

Another Numerical Example

- Now, buyers can be of ten types $\{1, 2, \ldots, 10\}$.
- the prob. mass function
  $f = (0.1, 0.15, 0.15, 0.15, 0.1, 0.1, 0.08, 0.07, 0.04, 0.06)$
- verify that $\nu(t)$ is monotone in t
- it has values $(-8, -3, -1, 1, 1.5, 3.5, 4.875, 6.571, 7.5, 10)$
- set up and solve the LP
- the optimal solution is $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = 0$
- $\mathcal{A}_4 = \mathcal{A}_5 = \ldots = \mathcal{A}_{10} = 1$
- $p_4 = p_5 = \ldots = p_{10} = 4$, $p_1 = p_2 = p_3 = 0$

The formulation II

- Introduce a dummy type $t = 0$ to hide the IR constraint ($\mathcal{A}_0 = p_0 = 0$):

$$\begin{aligned} \max \quad & \sum_{t=1}^{m} f_t p_t \\ \text{s.t.} \quad & t\mathcal{A}_t - p_t \geq t\mathcal{A}_s - p_s, \forall\ t, s \in T = \{0, 1, 2, \ldots, m\}\ (IC) \\ & \mathcal{A}_t \in [0, 1]\ \forall t \in T \end{aligned}$$

- We shall now digress to familiar territory!

Review of Shortest Paths

- Let $\mathcal{N}$ be the node-arc incidence matrix of a (directed) network $G = (V, A)$ with a single source node and sink node t
- Assume there is at least one $s - t$ path
- The incidence vector of a $s - t$ path corresponds to a flow of one unit through the network
- Hence finding a shortest $s - t$ path is equivalent to determining a minimum cost $s - t$ flow of one unit through the network
- Allows to define a polyhedron with all extreme points corresponding to $s - t$ paths.

The shortest path polyhedron

- Let $b^{s,t}$ be the vector such that
- $b_i^{s,t} = 0$ for all $i \in V \setminus \{s, t\}$
- $b_s^{s,t} = -1$
- $b_t^{s,t} = 1$
- The shortest path polyhedron is $\{x | \mathcal{N} x = b^{s,t}, x \geq 0\}$

The shortest path polyhedron II

- The following is well known:

  Every extreme point of $\{x | \mathcal{N}x = b^{s,t}, x \geq 0\}$ *is integral.*

- The following is also known (but less well known):

  Let $G = (V, A)$ *be network with source* s *and sink* t *and arc length vector* c. *A shortest* $s - t$ *path (wrt to* c*) exists if and only if* G *contains no negative length cycles*

  (I wonder who proved this first?)

- Now we shall connect the above to duality

The dual

- The shortest path problem has the dual:

$$\begin{aligned} \max \quad & y_t - y_s \\ \text{s.t.} \quad & \mathcal{N}^T y \leq c \end{aligned}$$

- The typical dual constraint:

$$y_j - y_i \leq c_{ij}, \forall\, (i,j) \in A$$

- Recall that one of the constraints in the primal is redundant
- Hence, set an arbitrary dual variable to zero, e.g., $y_s = 0$.

The dual II

- With $y_s = 0$ the dual becomes:

$$\max\{y_t | \mathcal{N}^T y \leq c, y_s = 0\}$$

- Let y^* denote an optimal solution
- By the duality theorem

  y_t^* is length of the shortest path form s to t

- Also

  for any other node i, y_i^* is length of the shortest path form s to i

- *For any feasible y with $y_s = 0$ y_i is bounded above by the length of the shortest $s - i$ path.*

The dual III

- Recall the duality theorem of LP one more time

  Primal is bounded iff dual is feasible

- Hence, we have by the previous development

  Dual is feasible iff network has no negative cycles

- These facts will be used
- Now, back to our direct mechanism

Back to the formulation

- Recall the formulation of the seller :

$$\begin{aligned} \max \quad & \sum_{t=1}^{m} f_t p_t \\ \text{s.t.} \quad & t\mathcal{A}_t - p_t \geq t\mathcal{A}_s - p_s, \forall\ t, s \in T = \{0, 1, 2, \ldots, m\}\ (IC) \\ & A_t \in [0, 1]\ \forall t \in T \end{aligned}$$

- Rewrite the constraint: $t\mathcal{A}_t - p_t \geq t\mathcal{A}_s - p_s$
- as the constraint

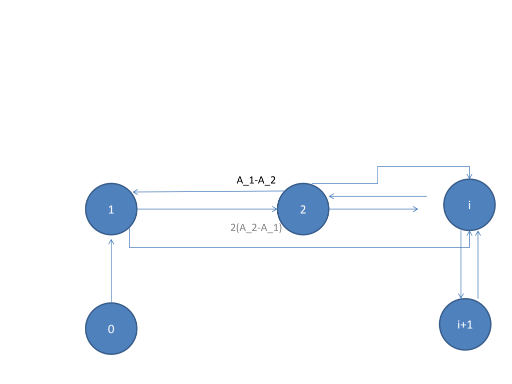
$$p_t - p_s \leq t(\mathcal{A}_t - \mathcal{A}_s)$$

- But this looks exactly like the dual shortest path constraints!

Incentive graph

- Indeed, make a graph
- with a node for each type,
- an arc for each ordered pair (s, t) with arc length $t(\mathcal{A}_t - \mathcal{A}_s)$
- for given $\mathcal{A}_t$, (IC) holds iff incentive graph has no negative cycles.
- So, our shortest path digression begins to pay off

The incentive graph



From incentive graph to algebra

- We need one more important observation

  No negative cycles in the incentive graph is equivalent to "$\mathcal{A}_t$ is monotone non-decreasing in t".

- Proof: add these two inequalities:

$$p_t - p_{t+1} \leq t(\mathcal{A}_t - \mathcal{A}_{t+1})$$

$$p_{t+1} - p_t \leq (t+1)(\mathcal{A}_{t+1} - \mathcal{A}_t)$$

- Also, recall that feasible p_t is upper bounded by length of shortest path to t
- Furthermore, at optimality we have $p_t = t\mathcal{A}_t - \sum_{j=1}^{t-1} \mathcal{A}_j$.

Transforming the problem

- Now, change the objective function using $p_t = t\mathcal{A}_t - \sum_{j=1}^{t-1} \mathcal{A}_j$
- We have

$$\sum_{t=1}^{m} f_t p_t = \sum_{t=1}^{m} f_t(t\mathcal{A}_t - \sum_{j=1}^{t-1} \mathcal{A}_t) = \sum_{t=1}^{m} f_t t\mathcal{A}_t - \sum_{t=1}^{m} \sum_{j=1}^{t-1} f_t \mathcal{A}_j$$

- Change the order of the summations in the second term

$$\sum_{t=1}^{m} f_t \sum_{j=1}^{t-1} \mathcal{A}_j = \sum_{t=1}^{m} \mathcal{A}_t (1 - F(t))$$

- Divide and multiply each term by f_t. The objective function:

$$\max \sum_{t=1}^{m} f_t \left(t - \frac{1 - F(t)}{f_t} \right) \mathcal{A}_t$$

Optimal solution

- Now, since the absence of negative cycles in the incentive graph is equivalent to the monotonicity of $\mathcal{A}_t$ we have the equivalent problem:

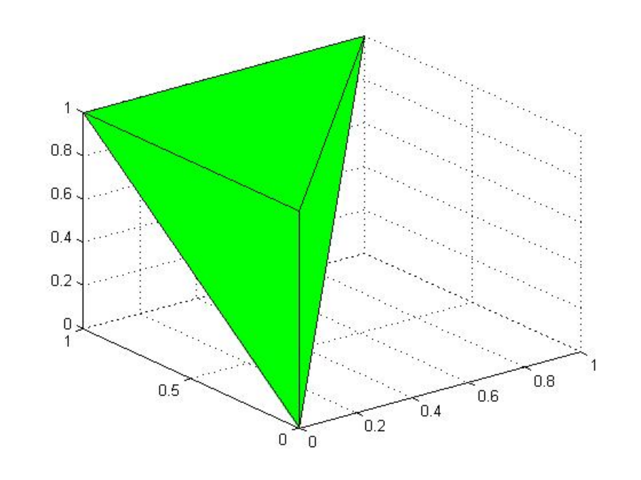
$$\max \sum_{t=1}^{m} f_t \left(t - \frac{1 - F(t)}{f_t} \right) \mathcal{A}_t$$

subject to

$$1 \geq \mathcal{A}_m \geq \mathcal{A}_{m-1} \geq \ldots \geq 0$$

- But this is solved immediately (recall that ν is monotone in t!):
- As long as $t - \frac{1-F(t)}{f_t} \geq 0$ set $\mathcal{A}_t = 1$!
- which is exactly what we observed in our example.

The monotone unit polytope: Monotope



Optimal solution

- But this is the simplest possible direct mechanism
- that is taught in elementary microeconomics
- set a price p and tell the buyer he can have the good if he is willing to pay the price p
- suppose the seller picks this procedure
- what price should he choose?
- Buyer will purchase if his type is at least as large as p. The probability of this event is $1 - F(p)$.
- Thus, expected revenue is $p(1 - F(p))$. Choose p to maximize this!
- Sufficient if $p(1 - F(p))$ is concave, i.e., $\nu(p)$ is non-decreasing!

Optimal mechanism

- First-order condition

$$p - \frac{1 - F(p)}{f_p} = 0$$

- sufficient if $p(1 - F(p))$ is concave, i.e. $t - \frac{1-F(t)}{f_t}$ is monotone non-decreasing.
- We obtained the discrete version of this.

Summary

- Let us review the assumptions made
- We have assumed a discrete type
- The analysis will go through with continuous types as well
- We have assumed risk-neutral buyer and seller
- The result can also be obtained with non-linear utilities.

Non-linear utility

- Consider selling an infinitely divisible good to a potential buyer
- it costs c per unit to the seller to produce the good
- the buyer has a concave utility function u
- we shall assume $u(x) = \sqrt{x}$
- All other assumptions are still valid

Non-linear utility

- The optimization problem giving the optimal direct mechanism

$$
\begin{aligned}
\max \quad & \sum_{t=1}^{m} f_t(p_t - c\mathcal{A}_t) \\
\text{s.t.} \quad & t\sqrt{\mathcal{A}_t} - p_t \ \geq \ t\sqrt{\mathcal{A}_s} - p_s, \forall\ t, s \in T\ (IC) \\
& t\sqrt{\mathcal{A}_t} - p_t \ \geq \ 0,\ \forall t \in T\ (IR) \\
& \mathcal{A}_t \ \in \ [0, \infty]\ \forall t \in T
\end{aligned}
$$

- Bad news: non-convex! But all is not lost.

Hidden convexity

- Consider a transformation: convex and equivalent to the previous problem

$$
\begin{aligned}
\max \quad & \sum_{t=1}^{m} f_t(p_t - c\mathcal{A}_t) \\
\text{s.t.} \quad & t\ y_t - p_t \ \geq\ t\ y_s - p_s, \forall\ t, s \in T\ (IC) \\
& t\ y_t - p_t \ \geq\ 0,\ \forall t \in T\ (IR) \\
& \sqrt{\mathcal{A}_t} \ \geq\ y_t,\ \forall t \in T \\
& \mathcal{A}_t \ \in\ [0, \infty]\ \forall t \in T \\
& y_t \ \in\ [0, \infty]\ \forall t \in T
\end{aligned}
$$

Simplification

- Using our shortest path trick the problem simplifies to

$$\begin{aligned} \max \quad & \sum_{t=1}^{m} f_t(\nu_t \sqrt{\mathcal{A}_t} - c\mathcal{A}_t) \\ \text{s.t.} \quad & \mathcal{A}_t \in [0, \infty] \; \forall t \in T \end{aligned}$$

- But this is easy to solve!

Structure of the Optimal Mechanism:

- The structure is similar to the previous case:
- $\mathcal{A}_t = p_t = 0$ for $t = 0, 1, \ldots, t^* - 1$
- $\mathcal{A}_t = (\frac{\nu_t}{2c})^2,\ p_t > 0$ for $t = t^*, \ldots, T$
- where t^* is the smallest t that satisfies:

$$\frac{1}{2}\frac{1}{\sqrt{\mathcal{A}_{t^*}}}\nu(t^*) = c$$

- optimal prices $p_t = t\ u(\mathcal{A}_t) - \sum_{j=1}^{t-1} u(\mathcal{A}_j)$.
- In general $\mathcal{A}_t = u^{-1}(\frac{\nu_t}{2c}),\ t = t^*, \ldots, T$

Robust(Stable) Direct Mechanisms

- Crucial assumption (**common prior**): distribution of types f known to the seller
- Let us assume the distribution is ambiguous around a reference probability mass $\bar{f}$:

$$\mathcal{P} = \{f | f \geq 0, e^T f = 1, \|f - \bar{f}\|_2 \leq \epsilon\}.$$

- We are interested in direct mechanisms that will maximize expected revenue under ambiguity of probability mass:

$$\max_{(p,\mathcal{A}) \in SP} \min_{f \in \mathcal{P}} p^T f$$

where SP are the (IC) constraints and simple bounds.

Robust(Stable) Direct Mechanisms

- Transform the max min problem (using conic duality):
- $$\max_{y,z,q,p,\mathcal{A}} y - \bar{f}^T q - \epsilon z$$

  subject to

  $$p + q \geq ye$$

  $$\|q\|_2 \leq z$$

  $$(p, \mathcal{A}) \in SP$$

- $e = (1, 1 \ldots, 1)^T$ is the vector of ones.

Robust(Stable) Direct Mechanisms

- Simplify the problem:
- $$\max_{y,q,p,\mathcal{A}} y - \bar{f}^T q - \epsilon \|q\|_2$$

  subject to

  $$p + q \geq ye$$

  $$(p, \mathcal{A}) \in SP$$
- At optimality $ye = p + q$ (easy to prove).

A Stability Result

- Further simplification :
- gives

$$\max_{y,p,\mathcal{A}} \bar{f}^T p - \epsilon \|ye - p\|_2$$

  subject to

$$(p, \mathcal{A}) \in SP$$

- *There exists* $\epsilon^* > 0$ *such that for* $\epsilon \in [0, \epsilon^*]$ *the optimal direct mechanism for* $\bar{f}$ *solves the above problem.*
- Proof: Using an old result of Mangasarian and Meyer (1978) *SIAM J. Control and Optim.*

In the Absence of a Reference Mass

- Consider the following set of discrete probability measures

$$\mathcal{I} = \{f | f \geq 0, e^T f = 1, \ell e \leq f \leq ue\}$$

for $\ell < u$ ($\ell \leq 1/m$)

- i.e., the seller can only predict the type probabilities up to an interval $[\ell, u]$
- The optimum allocation/payment mechanism is obtained from the optimal solution of

$$\max_{(p,\mathcal{A}) \in SP} \min_{f \in \mathcal{I}} p^T f$$

where SP are the (IC) constraints and simple bounds.

In the Absence of a Reference Mass

- The problem is transformed using LP duality into

$$\max_{\lambda, p, \mathcal{A}, y, z} \lambda + \ell e^T y - u e^T z$$

subject to

$$p - \lambda e = y - z$$

$$(p, \mathcal{A}) \in SP, \ (y, z) \in \mathbb{R}^m_+.$$

- an LP with a larger number of variables

Optimal Mechanism in the Absence of a Reference Mass

- Three things can happen:
- Either $\mathcal{A}_t^* = 1$, $p_t^* = 1$ for $t = 1, \ldots, m$
- Or $\mathcal{A}_t^* = 1$, $p_t^* = 2$ for $t = 2, \ldots, m$
- These are somewhat uninteresting!
- But, there is a third possibility.

Optimal Mechanism in the Absence of a Reference Mass

- Under certain conditions on m, ℓ, u (essentially, as m gets larger $u - \ell$ should follow suit, and *vice versa*)
- $\mathcal{A}_t^* = 1$ for $t = t^*, t^* + 1, \ldots, m$
- where $t^* = \lfloor \frac{m}{2} \pm 1 \rfloor$ or $\lceil \frac{m}{2} \pm 1 \rceil$
- $p_t^* = t^*$ for $t = t^*, t^* + 1, \ldots, m$; zero otherwise.
- E.g., for $m = 10$, $[\ell, u] = [0.01, 0.1]$, $t^* = 5$.
- for $m = 20$ $[\ell, u] = [0.01, 0.45]$, $t^* = 11$.
- (Joint work with C. Kizilkale).

Extensions

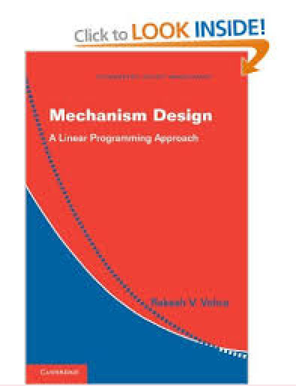
- There are many extensions, each more interesting than the previous one
- E.g., multiple agents for the sale of a single good
- in fact one can have multiple goods as well
- there can be a budget constraint for each agent
- the seller might inspect an agent's report of type for a cost (allocation with inspection)
- In all above cases, **max flow-min cut duality** and **polymatroid theory** are heavily used to determine the optimal mechanism!
- All these in a future seminar!

References

- R. Vohra, Optimization and mechanism design, *Math. Prog.* Series B, 2012, 134: 283–303.
- R. Vohra, *Mechanism design: a linear programming approach*, 2011, Cambridge University Press.
- T. Börgers, *Introduction to the theory of mechanism design*, 2008, Monograph, University of Michigan.

The man: Rakesh Vohra

The book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