A Note on Measures of Model Uncertainty and Calibrated Option Bounds

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Abstract

Recently Cont [3] introduced a quantitative framework for measuring model uncertainty in the context of derivative pricing. Two measures of model uncertainty were proposed: one measure based on a coherent risk measure compatible with market prices of derivatives, and another measure based on convex risk measures. We show that the two measures introduced by Cont [3] are closely related to calibrated option bounds studied recently by King *et al.* [7]. The precise relationship is established through convex programming duality.

Key words. Model uncertainty, option pricing, incomplete markets, coherent risk measures, convex risk measures, calibrated option bounds, duality.

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1 Introduction

In an arbitrage-free and complete financial market where the asset prices evolve according to some probability measure, the assumption of linearity of prices implies the existence of a unique equivalent measure such that the value of an option is computed as the expected value of its (discounted) pay-off under this equivalent measure that also makes asset prices into a martingale. However, this equivalent martingale measure is not uniquely specified when the market is incomplete. Therefore, one faces the issue of choosing an appropriate martingale measure among infinitely many possibilities in valuing a future stochastic pay-off. The problem of non-unique specification of a pricing rule is termed "model uncertainty" in Cont [3] which proposed two measures of model uncertainty satisfying certain requirements for quantifying ambiguity in the context of pricing a contingent claim. Against this background, the purpose of the present note is to present the relationship between measures of model uncertainty introduced by Cont [3] and calibrated option bounds studied recently by King *et*

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al. [7]. We show using convex duality theory that the two measures defined by Cont [3] are obtained directly from the calibrated option bounds. More precisely, the coherent measure of model uncertainty is obtained as the difference of the calibrated option bounds of King *et al.* Moreover, these bounds are computable as the optimal value of optimization problems corresponding to the hedging policies of a writer and a buyer of the contingent claim under study where the writer (and/or buyer) also includes a static (buy-and-hold) strategy using the benchmark options traded in the market in addition to trading in the underlying. For simplicity, the results are derived in a discrete time, finite probability space framework. A direct consequence of our results is that the second measure of model uncertainty of Cont [3], based on convex risk measures, yields a number at least as large as the first measure based on coherent risk measures.

The rest of this paper is organized as follows. In section 2 we review the model uncertainty and risk measures introduced by Cont [3]. We specify our market model in section 3, and we describe the calibrated option bounds as well as the precise connections between the previous section. Section 4 concludes with the convex risk measure-based model uncertainty.

2 Model Uncertainty and Coherent Risk Measures

Cont [3] introduced a methodology for measuring model uncertainty using the following ingredients:

1. Benchmark instruments: these are derivative instruments traded in the market with prices that can be observed. Let us denote the index set of available benchmark instruments by I (of cardinality K), their pay-offs with $(H^i)_{i\in I}$, and their observed market prices by $(C_i^*)_{i\in I}$. Typically, instead of a unique price, we have the bid and ask prices for buying and selling. Therefore we have $C_i^* \in [C_i^b, C_i^a]$.

2. A set of arbitrage-free pricing models $\mathcal{Q}$, i.e., a set of risk-neutral probability measures $\mathbb{Q}$ on some suitable set of market scenarios $(\Omega, \mathcal{F})$ consistent with the market prices of benchmark instruments with the property that the discounted underlying asset(s) prices $(S_t)_{t\in[0,T]}$ is a martingale under each $\mathbb{Q} \in \mathcal{Q}$ with respect to $\mathcal{F}_t$, and

$$\forall \mathbb{Q} \in \mathcal{Q}, \forall i \in I,\ \mathbb{E}^{\mathbb{Q}}[|H^i|] < \infty,\ \mathbb{E}^{\mathbb{Q}}[H^i] \in [C_i^b, C_i^a]. \tag{1}$$

Let us now define the set $\mathcal{C}$ of contingent claims with a well-defined price in all models:

$$\mathcal{C} = \{H \in \mathcal{F}_T, \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[|H|] < \infty\}. \tag{2}$$

Let $(\phi_t)_{t\in[0,T]}$ represent a self-financing trading strategy with the stochastic integral $\int_0^t \phi_u.dS_u$ corresponding to the (discounted) gain from trading between 0 and t. Consider now a mapping $\mu : \mathcal{C} \mapsto [0, \infty)$ representing the model uncertainty on a contingent claim X. Cont [3] imposes the following conditions on the model uncertainty measure μ:

1. For liquid benchmark instruments, model uncertainty reduces to the uncertainty in market prices:

$$\forall i \in I,\ \mu(H^i) \leq |C_i^a - C_i^b|. \tag{3}$$

2. Effect of hedging using the underlying asset(s):

$$\forall \phi \in \mathcal{S},\ \mu(X + \int_0^T \phi_t.dS_t) = \mu(X), \tag{4}$$

In particular, the value of a contingent claim which can be replicated by trading in the underlying has no model uncertainty.

$$[\exists x_0 \in \mathbb{R}, \exists \phi \in \mathcal{S}, \forall \mathbb{Q} \in \mathcal{Q},\ \mathbb{Q}(X = x_0 + \int_0^T \phi_t.dS_t) = 1] \Longrightarrow \mu(X) = 0. \tag{5}$$

3. Convexity: model uncertainty can be decreased through diversification.

$$\forall X_1, X_2 \in \mathcal{C},\ \forall \lambda \in [0,1]\ \mu(\lambda X_1 + (1-\lambda)X_2) \le \lambda\mu(X_1) + (1-\lambda)\mu(X_2). \tag{6}$$

4. Static hedging using benchmark instruments:

$$\forall X \in \mathcal{C},\ \forall u \in \mathbb{R}^K,\ \ \mu(X + \sum_{i=1}^K u_i H^i) \le \mu(X) + \sum_{i=1}^K |u_i(C_i^a - C_i^b)|. \tag{7}$$

In particular, if the pay-off can be statically replicated by benchmark derivatives then the model uncertainty measure has a value which is at most the cost of the static replication:

$$[\exists u \in \mathbb{R}^K, X = \sum_{i=1}^K u_i H^i] \Longrightarrow \mu(X) \le \sum_{i=1}^K |u_i|\ |C_i^a - C_i^b|. \tag{8}$$

Under the above requirements, the coherent measure of model uncertainty defined in Cont [3] is the following number

$$\mu_{\mathcal{Q}} = \overline{\pi}(X) - \underline{\pi}(X) \tag{9}$$

for $X \in \mathcal{C}$ and where

$$\overline{\pi}(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X],\ \ \underline{\pi}(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X]. \tag{10}$$

The mapping $X \mapsto \overline{\pi}(-X)$ defines a coherent risk measure in the sense of Föllmer and Schied [5].

The following was proved in Proposition 1 of [3].

Proposition 1 *1.* $\overline{\pi}, \underline{\pi}$ *assign values to the benchmark derivatives compatible with their market bid-ask prices:*

$$\forall i \in I,\ \ C_i^b \le \underline{\pi}(H^i) \le \overline{\pi}(H^i) \le C_i^a,$$

2. $\mu_{\mathcal{Q}} : C \mapsto \mathbb{R}^+$ *defined by (9) is a measure of model uncertainty verifying (3)-(4)-(5)- (6)-(7)-(8).*

3 Calibrated Option Bounds

Now, we adopt the setting of [7] by modeling security prices and other payments as discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms are sequences of real-valued vectors (asset values) over the discrete time periods $t = 0, 1, \dots, T$. We further assume the market evolves as a discrete, non-recombinant scenario tree in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted $a(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$ has a non-empty set of child nodes $\mathcal{D}(n) \subset \mathcal{N}_{t+1}$. The uniqueness of the parent node makes the scenario tree non-recombinant, an essential feature in specifying incomplete market models [4]. We denote the set of all nodes in the tree by $\mathcal{N}$. The probability distribution P is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-terminal (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m \in \mathcal{D}(n)} p_m, \;\; \forall\, n \in \mathcal{N}_t, \; t = T-1, \ldots, 0.$$

Hence, each intermediate node has a probability mass equal to the combined mass of the paths passing through it. The ratios $p_m/p_n, m \in \mathcal{D}(n)$ are the conditional probabilities that the child node m is visited given that the parent node $n = a(m)$ has been visited. We note that no particular form is assumed for P, i.e., the price process could have jumps, it could be non-Markovian, or it may incorporate stochastic volatility.

A random variable X is a real valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω, [6]. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^P[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is given by the expression

$$\mathbb{E}^P[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{D}(n)} \frac{p_m}{p_n} X_m.$$

Under the light of the above definitions, the market consists of $J+1$ market-traded securities indexed by $j = 0, 1, \ldots, J$ with prices at node n given by the vector $S_n = (S_n^0, S_n^1, \ldots, S_n^J)$. We assume that the security indexed by 0 has strictly positive prices at each node of the scenario tree. This asset corresponds to the risk-free asset in the classical valuation framework. Choosing this security as the numéraire, we can scale the prices at each node we obtain $S_n^0 = 1$ for all nodes $n \in \mathcal{N}$. For the sake of simplicity, we will assume that the prices have already been scaled with respect to the numéraire.

The amount of security j held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted θ_n^j. Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$. The value of the portfolio at state n (discounted with respect to the numéraire) is

$$S_n \cdot \theta_n = \sum_{j=0}^{J} S_n^j \theta_n^j.$$

We will say that the vector process $\{S_t\}$ is called a vector-valued martingale under Q, and that Q is called a martingale probability measure for the process if there exists a probability measure $Q = \{q_n\}_{n\in\mathcal{N}_T}$ such that

$$S_t = \mathbb{E}^Q[S_{t+1}|\mathcal{N}_t] \ (t \leq T-1). \tag{11}$$

By a contingent claim we mean a stochastic cash-flow $F \in \mathcal{C}$ which in our present setting is characterized by (discounted) pay-outs $\{F_n\}_{n\in\mathcal{N}}$ that depend on the price process S of the underlying securities. King *et al.* [7] formulate the problem of the *writer* of the contingent claim F as computing the smallest amount of initial cash outlay required to hedge the pay-outs generated by the contingent claim by self-financing transactions so as to end up with a non-negative wealth position almost surely at the expiry date of the contingent claim. This initial cash outlay is the optimal value of the optimization problem

$$\begin{array}{lrcl} \min & V & & \\ \text{s.t.} & S_0 \cdot \theta_0 & = & V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) & = & -F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n & \geq & 0, \ \forall\, n \in \mathcal{N}_T. \end{array}$$

When there are other options (benchmark derivatives) available for trading and they are used for static hedging purposes in the above model, one obtains the writer's problem (WC):

$$\begin{array}{lrcl} \min & V & & \\ \text{s.t.} & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- & = & V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) & = & H_n \cdot (\xi_+ - \xi_-) - F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n & \geq & 0, \ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- & \geq & 0, \end{array}$$

where $H^k, k = 1, \ldots, K$ represent the benchmark derivatives with bid-ask prices C^b_k and C^a_k, and (already discounted) pay-offs H^k_n, for all $n \in \mathcal{N}$ (i.e., H_n is a K-vector for all n), and the vectors $\xi_+, \xi_- \in \mathbb{R}^K$ are the amounts bought and shorted of each benchmark derivative instrument. Denote the optimal value in this problem by $V_w(F)$.

The hedging strategy of the buyer, which is the opposite of the writer, is obtained from the optimal solution of the following problem (BC):

$$\begin{array}{lrcl} \max & V & & \\ \text{s.t.} & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- & = & -V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) & = & H_n \cdot (\xi_+ - \xi_-) + F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n & \geq & 0, \ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- & \geq & 0. \end{array}$$

Denote the optimal value of the above problem by $V_b(F)$.

The numbers $V_w(F)$ and $V_b(F)$ correspond to the *calibrated option bounds* that originated in Avellaneda *et al.* [1, 2], and further developed in King *et al.* [7]. In this approach to computing bounds for option prices, market-traded options are used in the trading strategies of the seller and the buyer resulting in price measures (pricing rules) that are consistent with the observed market prices exactly as advocated in the previous section for the measure of model uncertainty. Therefore, our first observation is the following.

Proposition 2 *For each $F \in \mathcal{C}$, we have $\mu_{\mathcal{Q}}(F) = V_w(F) - V_b(F)$.*

Proof: From [7], the dual of (WC) is the following linear programming problem in variables $\{y_n\}_{n\in\mathcal{N}}$:

$$\begin{aligned}
\max \quad & \sum_{t=1}^{T}\sum_{n\in\mathcal{N}_t} y_n F_n \\
\text{s.t.} \quad & y_0 = 1 \\
& y_m S_m = \sum_{n\in\mathcal{D}(m)} y_n S_n, \ \forall\, m \in \mathcal{N}_t, 0 \le t \le T-1 \\
& \sum_{t=1}^{T}\sum_{n\in\mathcal{N}_t} y_n H_n \le C^a \\
& \sum_{t=1}^{T}\sum_{n\in\mathcal{N}_t} y_n H_n \ge C^b \\
& y_n \ge 0 \ \forall n \in \mathcal{N}_t.
\end{aligned}$$

By Theorem 4.1 of [7], the dual problem is equivalently expressed as

$$\sup_{Q\in\mathcal{M}_C} \mathbb{E}^Q[\sum_{t=1}^{T} F_t]$$

where $\mathcal{M}_C = \{Q \in \mathcal{M} | C^b \le \mathbb{E}^Q[\sum_{t=1}^{T} H_t] \le C^a\}$ with $\mathcal{M}$ denoting the set of all martingale probability measures (not necessarily equivalent to P), i.e., the set of all q_n, $n \in \mathcal{N}$ satisfying

$$q_n \ge 0, \ n \in \mathcal{N}_T,$$

$$q_n S_n = \sum_{m\in D(n)} q_m S_m, \ n \in \mathcal{N}_t, t \le T-1,$$

$$q_0 = 1$$

(c.f. Proposition 1 of [6]). Therefore, in our finite probability space, discrete time setting $\mathcal{M}_C$ and $\mathcal{Q}$ coincide. ∎

Note that both problems WC and BC involved in computing $\mu_{\mathcal{Q}}$ are linear programming problems that can be routinely solved using available software.

4 A Convex Measure of Model Uncertainty

In [3], a measure of uncertainty based on convex risk measures in the sense of Föllmer and Schied [5] was also introduced. However, an important difference is that $\mathcal{Q}$ no longer represents a set of pricing rules consistent with the prices of benchmark instruments. Instead, it is assumed to contain all measures that make the underlying asset prices a martingale. Notice that $\mathcal{Q}'$ coincides with the set $\mathcal{M}$ in the proof of Proposition 2 in our finite state probability and discrete time context.

This second measure of model uncertainty is defined using

$$\pi^*(X) = \sup_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] - \|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_p\} \tag{12}$$

$$\pi_*(X) = \inf_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] + \|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_p\} \tag{13}$$

assuming a unique price vector C^* for the benchmark instruments, and where $\|z\|_p = \sqrt[p]{\sum_{i=1}^n |z_i|^p}$ for some $z \in \mathbb{R}^K$ for $1 < p < \infty$. For $p = 1, \infty$, we deal with the penalty terms $\|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_1 = \sum_{i=1}^K |C_i^* - \mathbb{E}^{\mathbb{Q}}[H^i]|$, and $\|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_\infty = \max_{i=1,\ldots,K} |C_i^* - \mathbb{E}^{\mathbb{Q}}[H^i]|$, respectively. Then, the model uncertainty measure is defined as

$$\forall X \in \mathcal{C}, \;\; \mu_*^p(X) = \pi^*(X) - \pi_*(X). \tag{14}$$

Allowing for bid and ask prices, the associated bounds are defined as:

$$\pi^*(X) = \sup_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] - \|(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p - \|(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\} \tag{15}$$

and

$$\pi_*(X) = \inf_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] + \|(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p + \|(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\}, \tag{16}$$

where the operator $(.)_+ = \max\{0, .\}$ is applied to each component of the vectors $\mathbb{E}^{\mathbb{Q}}[H] - C^a$ and $C^b - \mathbb{E}^{\mathbb{Q}}[H]$. Instead of calibrating the martingale measure according to bid-ask prices of the benchmark instruments, the last two terms involving norms in the definition of the bounds above penalize deviations from bid-ask prices of the benchmark options. In the language of Föllmer and Schied $\rho(X) = \pi^*(-X)$ is a convex risk measure; see [5, 3]. Under some suitable assumptions including one that imposes that the set $\mathcal{Q}'$ contain a least one measure $\mathbb{Q}$ that gives

$$\mathbb{E}^{\mathbb{Q}}[H^i] \in [C_i^b, C_i^a] \; \forall i \in I,$$

Cont [3] proves that the model uncertainty measure μ_* satisfies (3)-(4)-(5)-(6), and the appropriate modifications of (7) and (8); see Proposition 2 of [3] and the discussion therein.

Let us now fix some q such that $1 \leq q \leq \infty$, and consider in the discrete time, finite state framework of calibrated option bounds the following writer's optimal hedging problem CWC:

$$\begin{array}{lrcl} \inf & V & & \\ \text{s.t.} & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- & = & V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) & = & H_n \cdot (\xi_+ - \xi_-) - F_n,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n & \geq & 0,\ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- & \geq & 0, \\ & \|\xi_+\|_q & \leq & 1, \\ & \|\xi_-\|_q & \leq & 1, \end{array}$$

with optimal value $VC_w^q(F)$, and the buyer's hedging problem CBC

$$\begin{array}{lrcl} \sup & V & & \\ \text{s.t.} & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- & = & -V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) & = & H_n \cdot (\xi_+ - \xi_-) + F_n,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n & \geq & 0,\ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- & \geq & 0, \\ & \|\xi_+\|_q & \leq & 1, \\ & \|\xi_-\|_q & \leq & 1, \end{array}$$

with optimal value $VC_b^q(F)$. We notice that the above optimization problems are almost identical to those of the previous section with the additional restriction that the long and short static hedge positions in traded (benchmark) options are bounded in some suitable norm. This is related to Stockbridge [9] which considers the superhedging problem for option pricing while limiting the short positions in the underlying and the bond. This reference gives a stochastic process interpretation of the resulting dual as well.

Now, we can state the following observation.

Proposition 3 *For $F \in \mathcal{C}$, and $1 \leq q \leq \infty$ we have*

1. $\mu_*^p(F) = VC_w^q(F) - VC_b^q(F)$,
2. $\mu_{\mathcal{Q}}(F) \leq \mu_*^p(F)$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Using Lagrange duality it is immediate to verify that the convex programming dual of CWC is given by

$$\sup_{Q \in \mathcal{M}} \{\mathbb{E}^Q[\sum_{t=1}^{T} F_t] - \|(\mathbb{E}^Q[H] - C^a)_+\|_p - \|(C^b - \mathbb{E}^Q[H])_+\|_p\} \tag{17}$$

and that of BC is given as

$$\inf_{Q \in \mathcal{M}} \{\mathbb{E}^Q[\sum_{t=1}^{T} F_t] + \|(\mathbb{E}^Q[H] - C^a)_+\|_p + \|(C^b - \mathbb{E}^Q[H])_+\|_p\} \tag{18}$$

where $\mathbb{E}^Q[H]$ is a K-vector with the i-th component equal to $\mathbb{E}^Q[\sum_{t=1}^T H_t^i]$. The easiest way to see this duality relation is to re-write e.g. (17) first as

$$\sup_{Q\in\mathcal{M}}\{\mathbb{E}^Q[\sum_{t=1}^T F_t]+\inf_{\|\xi_+\|_q\le 1,\xi_+\ge 0}\xi_+^T(C^a-\mathbb{E}^Q[H])+\inf_{\|\xi_-\|_q\le 1,\xi_-\ge 0}\xi_-^T(\mathbb{E}^Q[H]-C^b)\}$$

using a dual representation of norms where $1/p+1/q=1$ (the non-negativity of ξ_+,ξ_- arises due to the $(.)_+$ operator). This is equivalent to

$$\sup_{Q\in\mathcal{M}}\inf_{\|\xi_+\|_q\le 1,\xi_+\ge 0,\|\xi_-\|_q\le 1,\xi_-\ge 0}\{\mathbb{E}^Q[\sum_{t=1}^T F_t]+\xi_+^T(C^a-\mathbb{E}^Q[H])+\xi_-^T(\mathbb{E}^Q[H]-C^b)\}.$$

Using Corollary 37.3.2 of [8] we can now exchange inf and sup since the set on which inf is taken is bounded, i.e., the previous expression is equal to

$$\inf_{\|\xi_+\|_q\le 1,\xi_+\ge 0,\|\xi_-\|_q\le 1,\xi_-\ge 0}\sup_{Q\in\mathcal{M}}\{\mathbb{E}^Q[\sum_{t=1}^T F_t]+\xi_+^T(C^a-\mathbb{E}^Q[H])+\xi_-^T(\mathbb{E}^Q[H]-C^b)\}.$$

Now, recalling the polyhedral description of $\mathcal{M}$ from the proof of Proposition 2, and proceeding to evaluate the inner sup using linear programming duality, one obtains the dual (or, primal) problem CWC. For the other bound, one writes (18) equivalently as

$$\inf_{Q\in\mathcal{M}}\{\mathbb{E}^Q[\sum_{t=1}^T F_t]+\sup_{\|\xi_+\|_q\le 1,\xi_+\ge 0}\xi_+^T(\mathbb{E}^Q[H]-C^a)+\sup_{\|\xi_-\|_q\le 1,\xi_-\ge 0}\xi_-^T(C^b-\mathbb{E}^Q[H])\}.$$

The rest of the argument is similar to the one above and leads to CBC as the dual problem. This proves part 1. Part 2 now follows from the observation that the problems CWC and CBC are more tightly constrained compared to their counterparts of section 3. ∎

Notice that for the typical choices of the norm, e.g., for $p=1$ and $p=\infty$ the writer's hedging problem becomes polyhedral convex programs:

$$\begin{aligned}
\inf\quad & V\\
\text{s.t.}\quad & S_0\cdot\theta_0+C^a\cdot\xi_+-C^b\cdot\xi_- = V\\
& S_n\cdot(\theta_n-\theta_{a(n)}) = H_n\cdot(\xi_+-\xi_-)-F_n,\ \forall\, n\in\mathcal{N}_t, t\ge 1\\
& S_n\cdot\theta_n \ge 0,\ \forall\, n\in\mathcal{N}_T,\\
& \xi_+,\xi_- \ge 0,\\
& \|\xi_+\|_\infty \le 1,\\
& \|\xi_-\|_\infty \le 1,
\end{aligned}$$

which is reminiscent of the Stockbridge [9] superhedging problem with finite limits on borrowing and shorting, and

$$\begin{aligned} \inf \quad & V \\ \text{s.t.} \quad & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) - F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- \geq 0, \\ & \|\xi_+\|_1 \leq 1, \\ & \|\xi_-\|_1 \leq 1. \end{aligned}$$

Both problems above can be transformed to linear programming problems. For the case $p = 2$, we are facing the convex programming problem with Euclidean unit-ball restrictions:

$$\begin{aligned} \inf \quad & V \\ \text{s.t.} \quad & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) - F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- \geq 0, \\ & \|\xi_+\|_2 \leq 1, \\ & \|\xi_-\|_2 \leq 1. \end{aligned}$$

All three problems above are efficiently processed using available optimization methods and software.

A variation on this theme is to consider weighted versions of the penalty terms in the definition of bounds

$$\pi^*(X) = \sup_{\mathbb{Q} \in \mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] - \|W(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p - \|W(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\} \tag{19}$$

and

$$\pi_*(X) = \inf_{\mathbb{Q} \in \mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] + \|W(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p + \|W(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\}, \tag{20}$$

where W is $K \times K$ diagonal matrix with positive diagonal entries; see section 5 of [3] for a discussion. In this case, the dual problems are simply modified as

$$\begin{aligned} \inf \quad & V \\ \text{s.t.} \quad & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) - F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- \geq 0, \\ & \|W^{-1}\xi_+\|_q \leq 1, \\ & \|W^{-1}\xi_-\|_q \leq 1, \end{aligned}$$

for π^*, and

$$\begin{array}{lrcl} \sup & V & & \\ \text{s.t.} & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- & = & -V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) & = & H_n \cdot (\xi_+ - \xi_-) + F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n & \geq & 0, \ \forall\, n \in \mathcal{N}_T, \\ & \xi_+, \xi_- & \geq & 0, \\ & \|W^{-1}\xi_+\|_q & \leq & 1, \\ & \|W^{-1}\xi_-\|_q & \leq & 1 \end{array}$$

for π_*.

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