Expected Gain-Loss Pricing and Hedging of Contingent Claims in Incomplete Markets by Linear Programming

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Abstract

We analyze the problem of pricing and hedging contingent claims in the multi-period, discrete time, discrete state case using the concept of a sufficiently attractive expected gain opportunity to a claim's writer and buyer. Pricing results somewhat different from, but reminiscent of, the arbitrage pricing theorems of mathematical finance are obtained. We show that our analysis provides tighter price bounds on the contingent claim in an incomplete market, which may converge to a unique price for a specific value of a risk aversion parameter imposed by the market while the hedging policies may be different for different sides of the same trade. The results are obtained in the simpler framework of stochastic linear programming in a multi-period setting, and have the appealing feature of being very simple to derive and to articulate even for the non-specialist.

Key words. Contingent claim, pricing, hedging, martingales, stochastic linear programming.

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1 Introduction

An important class of pricing theories in financial economics are derived under no-arbitrage conditions. When markets are complete, these theories yield unique sets of prices without any assumptions about individual investor's preferences. In other words, the pricing of assets relies on the availability and the liquidity of traded assets that span the full set of possible future states. Ross [20, 21] proves that the no-arbitrage condition is equivalent to the existence of a linear pricing rule and positive state prices that correctly value all assets. This linear pricing rule is the risk neutral probability measure in the Cox-Ross option pricing model, for example Harrison and Kreps [9] showed that the linear pricing operator is an expectation taken with respect to a martingale measure. However, when markets are incomplete state prices and claim prices are not unique. Since markets are almost never complete due

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to market imperfections as discussed in Carr *et al.* [4], and characterizing all possible future states of economy is impossible, alternative incomplete pricing theories have been developed.

A second class of pricing theories relies on the Expected Utility Hypothesis and requires the specification of investor preferences. This model equates the price of a claim to the expectation of the product of the future payoff and the marginal rate of substitution of the representative investor. Although this class of models has strong theoretical appeal (see e.g. [6, 11, 14] for related recent work), the difficulties in specifying preferences, especially for the representative agent, makes them impractical for pricing. Recent papers by Cochrane and Saa-Requejo [7], Bernardo and Ledoit [1], Carr *et al.* [4] and Roorda *et al.* [19] and Kallsen [14] unify these two classes of pricing theories and value options in an incomplete market setting.

The goal of this paper is to study the problem of pricing and hedging contingent claims in a general, multi-period, stochastic linear optimization (discrete-time, finite probability space) framework. Our analysis is based on a concept that we call a "sufficiently attractive expected gain" (SAGE, for short) opportunity that can accommodate complex pricing problems in a simple fashion. The motivations for this extension are twofold. The first motivation comes from the desire to obtain tighter bounds on a contingent claim's price compared with the arbitrage-free price bounds in incomplete markets, which will be a useful and practical tool for professionals. The point of departure here is that, while an arbitrage opportunity may not exist, the price system in a financial market may offer a SAGE. Therefore, investors might be engaging in buying and selling contingent claims with the aim of realizing a SAGE, if not an arbitrage. The second motivation, which is more of a technical nature, is that replacing the no-arbitrage arguments by a no-SAGE argument does not essentially change the linear pricing rule while it yields some new results briefly reviewed in the paragraphs below.

The replacement of the no-arbitrage conditions with a SAGE is similar to Bernardo and Ledoit [1] where they introduce the expected gain to loss ratio for pricing options in a single period, incomplete market framework. They obtain a duality theorem for maximization of the expected gain to loss ratio, which they use for establishing bounds on option prices. Our concept of sufficiently attractive expected gain is also similar to the notion of a good-deal that was developed in a series of papers by various authors [5, 7, 13, 22]. For example in Cochrane and Saa-Requejo [7], the absence of arbitrage is replaced by the concept of a good deal, defined as an investment with a high Sharpe ratio. While they do not use the term "good-deal", Bernardo and Ledoit [1] replace the high Sharpe ratio by the gain-loss ratio. These earlier studies are generalized using duality theory in infinite dimensional spaces in [5, 13, 22] but usually in single period models. To avoid confusing the reader with different definitions of the same term (good-deal) we introduced our own terminology of sufficiently attractive expected gains (SAGE) depending on a risk aversion (preference) parameter. Our definitions of SAGE are not based on ratios, thus leading to optimization problems that are easier to analyze.

Our results show that the market may actually arrive at a consensus about the pricing rule. However, in the incomplete market setting, the same pricing rule leads to different hedging policies for different sides of the same trade. This is an important finding as it will result in different demand and supply schemes for the replicating assets. Whereas the treatment of the afore-mentioned sources are usually in a more sophisticated mathematical setting, our results have the appealing feature of being very simple to derive and to articulate even for the non-specialist. While we stay in the much simpler setting of discrete time, discrete state stochastic programming developed by King [15], our analysis goes beyond [1] in the sense that we model multi-period investment problems. Stochastic programming offers a propitious framework for adding additional variables and constraints into the models as well as possibility of efficient numerical processing; see the book [2] for a thorough introduction to stochastic programming.

It is well-known (and established in [15] using simple arguments based on linear programming duality) that a price system in a financial market is arbitrage-free if and only if there exists an equivalent probability measure that makes the random price system a martingale. However, the existence of an equivalent martingale measure is not enough to guarantee that the system is SAGE-free. Therefore, one needs a special condition on the equivalent martingale measure to guarantee the non-existence of a SAGE opportunity. Once the risk aversion parameter is set we can guarantee that the price process is SAGE-free if there exists an equivalent martingale measure with an additional restriction. This sufficient condition is also necessary. Hence, we obtain the equivalent of the fundamental characterization of no arbitrage in the more general setting of sufficiently attractive expected gains. Furthermore, our construction leads to tighter bounds on the minimum price at which the writer of a contingent claim is ready to sell, and the maximum price at which a potential buyer is ready to acquire the contingent claim. These bounds converge to the no-arbitrage bounds in the limit as a risk aversion parameter goes to infinity (and hence, the investor essentially looks for an arbitrage). On the other extreme, as the risk aversion parameter goes down to the smallest value not allowing a sufficiently attractive expected gain opportunity, the writer and buyer's SAGE-free prices of a contingent claim may converge to a single value, hence potentially providing a unique price for the contingent claim in an incomplete market.

The organization of the paper is as follows. In section 2 we review the stochastic process governing the asset prices and we lay out the basics of our analysis. Section 3 gives an exposition of our first results in the context of sufficiently attractive expected gains based on the gain-loss trade-off. We consider a related problem in section 4 where the investor in search of a SAGE would also like to find the SAGE with the largest possible value of the preference (risk aversion) parameter controlling the SAGE. Here we re-obtain a duality result which turns out be essentially the duality result of Bernardo and Ledoit in a multi-period but finite probability state space setting. In section 5 we analyze the pricing problems of writers and buyers of contingent claims under the SAGE viewpoint. Proportional transaction costs are considered in section 6. Section 7 presents an extension of the SAGE concept incorporating the Conditional Value-at-Risk measure. We use simple numerical examples to illustrate our results. The paper is concluded in section 8.

2 The Stochastic Scenario Tree, Arbitrage and Martingales

Throughout this paper we follow the general probabilistic setting of [15] in that we approximate the behavior of the stock market by assuming that security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms ω are sequences of real-valued vectors (asset values) over the discrete time periods $t = 0, 1, \ldots, T$. We further assume the market evolves as a discrete, non-recombinant scenario tree (hence, suitable for incomplete markets) in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted $\pi(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$ has a non-empty set of child nodes $\mathcal{S}(n) \subset \mathcal{N}_{t+1}$. We denote the set of all nodes in the tree by $\mathcal{N}$. The probability distribution P is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-terminal (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m \in \mathcal{S}(n)} p_m, \;\; \forall\, n \in \mathcal{N}_t, \; t = T-1, \ldots, 0.$$

Hence, each intermediate node has a probability mass equal to the combined mass of the paths passing through it. The ratios $p_m/p_n, m \in \mathcal{S}(n)$ are the conditional probabilities that the child node m is visited given that the parent node $n = \pi(m)$ has been visited. This setting is chosen as it accommodates multi-period pricing for future different states and time periods at the same time, employing realization paths in the valuation process. It is a general framework that allows to address the valuation problem with incomplete markets and heterogeneous beliefs which are very stringent assumptions in the classical valuation theory. In this respect, it improves our understanding of valuation in a simple, yet complete fashion.

A random variable X is a real valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω, [15]. This kind of random variable is said to be measurable with respect to the information contained in the nodes of $\mathcal{N}_t$. A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is measurable with respect $\mathcal{N}_t$. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^P[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is given by the expression

$$\mathbb{E}^P[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{S}(n)} \frac{p_m}{p_n} X_m.$$

Note that this conditional expectation is a random variable taking values of the nodes $n \in \mathcal{N}_t$. Under the light of the above definitions, the market consists of $J+1$ tradable securities indexed by $j = 0, 1, \ldots, J$ with prices at node n given by the vector $S_n = (S_n^0, S_n^1, \ldots, S_n^J)$. We assume as in [15] that the security indexed by 0 has strictly positive prices at each node of the scenario tree. This asset corresponds to the risk-free asset in the classical valuation framework. Choosing this security as the numéraire, and using the discount factors $\beta_n = 1/S_n^0$ we define $Z_n^j = \beta_n S_n^j$ for $j = 0, 1, \ldots, J$

and $n \in \mathcal{N}$, the security prices discounted with respect to the numéraire. Note that $Z_n^0 = 1$ for all nodes $n \in \mathcal{N}$.

The amount of security j held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted θ_n^j. Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$. We refer to the collection of vectors $\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|}$ as Θ. The value of the portfolio at state n (discounted with respect to the numéraire) is

$$Z_n \cdot \theta_n = \sum_{j=0}^{J} Z_n^j \theta_n^j.$$

We will work with the following definition of arbitrage: an arbitrage is a sequence of portfolio holdings that begins with a zero initial value (note that short sales are allowed), makes self-financing portfolio transactions throughout the planning horizon and achieves a non-negative terminal value in each state, while in at least one terminal state it achieves a positive value with non-zero probability. The self-financing transactions condition is expressed as

$$Z_n \cdot \theta_n = Z_n \cdot \theta_{\pi(n)},\ n > 0.$$

The stochastic programming problem used to seek an arbitrage is the following optimization problem (P1):

$$\begin{array}{lrcl} \max & \displaystyle\sum_{n \in \mathcal{N}_T} p_n Z_n \cdot \theta_n & & \\ \text{s.t.} & Z_0 \cdot \theta_0 & = & 0 \\ & Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n & \geq & 0,\ \forall\, n \in \mathcal{N}_T. \end{array}$$

If there exists an optimal solution (i.e., a sequence of $\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|}$ vectors) which achieves a positive optimal value, this solution can be turned into an arbitrage as demonstrated by Harrison and Pliska [10].

We need the following definitions.

Definition 1 *If there exists a probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ such that*

$$Z_t = \mathbb{E}^Q[Z_{t+1} | \mathcal{N}_t]\ (t \leq T-1) \tag{1}$$

then the vector process $\{Z_t\}$ is called a vector-valued martingale under Q, and Q is called a martingale probability measure for the process. If one has coordinate-wise $Z_t \geq \mathbb{E}^Q[Z_{t+1}|\mathcal{N}_t], (t \leq T-1)$ (respectively, $Z_t \leq \mathbb{E}^Q[Z_{t+1}|\mathcal{N}_t], (t \leq T-1)$ the process is called a super-martingale (sub-martingale, respectively).

Definition 2 *A discrete probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ is equivalent to a (discrete) probability measure $P = \{p_n\}_{n \in \mathcal{N}_T}$ if $q_n > 0$ exactly when $p_n > 0$.*

King proved the following (c.f. Theorem 1 of [15]):

Theorem 1 *The discrete state stochastic vector process $\{Z_t\}$ is an-arbitrage free market price process if and only if there is at least one probability measure Q equivalent to P under which $\{Z_t\}$ is a martingale.*

The above result is the equivalent of Theorem 1 of Harrison and Kreps [9] in our setting.

3 Sufficiently Attractive Expected Gain and Martingales

In our context a sufficiently attractive expected gain opportunity is defined as follows. For $n \in \mathcal{N}_T$ let $Z_n \cdot \theta_n = x_n^+ - x_n^-$ where x_n^+ and x_n^- are non-negative numbers, i.e., we express the final portfolio value at terminal state n as the difference of two non-negative numbers. Assume that there exist a set of vectors $\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|}$ such that

$$Z_0 \cdot \theta_0 = 0$$

$$Z_n \cdot (\theta_n - \theta_{\pi(n)}) = 0, \; \forall\, n \in \mathcal{N}_t, t \geq 1$$

and

$$\mathbb{E}^P[X^+] - \lambda \mathbb{E}^P[X^-] > 0,$$

for $\lambda > 1$, where $X^+ = \{x_n^+\}_{n \in \mathcal{N}_T}$, and $X^- = \{x_n^-\}_{n \in \mathcal{N}_T}$. This sequence of portfolio holdings is said to yield a sufficiently attractive expected gain opportunity at level $\lambda > 1$. This formulation is similar to Bernardo and Ledoit [1] gain-loss ratio, and the Sharpe ratio restriction of Cochrane and Saa-Requejo [7]. Yet, it makes the problem easier to tackle within the framework of linear programming. Moreover, the parameter λ can be interpreted as the risk aversion parameter of the individual investor. As λ gets bigger, the individual will become more risk-averse, preferring near-arbitrage positions. As λ gets closer to 1, the individual weighs the gains and losses equally. In the limiting case of λ being equal to 1 the pricing operator (equivalent martingale measure) is unique if it exists. In fact, the pricing operator may become unique at a value of λ larger than one, which is what we expect in a typical pricing problem.

Consider now the perspective of an investor who is content with the existence of a sufficiently attractive expected gain (SAGE) opportunity although an arbitrage opportunity does not exist. Such an investor is interested in the solution of the following stochastic linear programming problem that we refer to as (SP1):

$$\begin{array}{lrcl}
\max & \displaystyle\sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- & & \\
\text{s.t.} & Z_0 \cdot \theta_0 & = & 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0, \; \forall\, n \in \mathcal{N}_T, \\
& x_n^+ & \geq & 0, \; \forall\, n \in \mathcal{N}_T, \\
& x_n^- & \geq & 0, \; \forall\, n \in \mathcal{N}_T.
\end{array}$$

If there exists an optimal solution (i.e., a sequence of $\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|}$ vectors) to the above problem that yields a positive optimal value, the solution is said to give rise to a sufficiently attractive expected gain opportunity at level λ (the expected positive terminal wealth outweighing λ times the expected negative final wealth). If there exists a SAGE opportunity in SP1, then SP1 is unbounded. We note that by the fundamental theorem of linear programming, when it is solvable, SP1 has always a basic optimal solution in which no pair x_n^+, x_n^-, for all $n \in \mathcal{N}_t$, can be positive at the same time.

We will say that the discrete state stochastic vector process $\{Z_t\}$ is a SAGE-free market price process at a fixed level λ if the optimal value of the above stochastic linear program is equal to zero. Clearly, if λ tends to infinity we essentially recover King's problem P1. It is a well-accepted phenomenon that every rational investor is ready to lose if the benefits of the gains outweigh the costs of the losses. It is also reasonable to assume that the rational investor will try to limit losses. This type of behavior excluded by the no-arbitrage setting is easily modeled by the Expected Utility approach. Our formulation allows investors to take reasonable risks without explicitly specifying a complicated utility function while it converges to the no-arbitrage setting in the limit. It is easy to see that an arbitrage opportunity is also a SAGE opportunity, and that absence of a SAGE opportunity (at any level λ) implies absence of arbitrage. It follows from Theorem 1 that if the market price process is SAGE-free at a level λ then there exists an equivalent measure that makes the price process a martingale.

Definition 3 *Given $\lambda > 1$ a discrete probability measure $Q = \{q_n\}_{n\in\mathcal{N}_T}$ is λ-compatible to a (discrete) probability measure $P = \{p_n\}_{n\in\mathcal{N}_T}$ if it is equivalent to P (Definition 2) and satisfies*

$$\max_{n\in\mathcal{N}_T} p_n/q_n \leq \lambda \min_{n\in\mathcal{N}_T} p_n/q_n.$$

Theorem 2 *The process $\{Z_t\}$ is a SAGE-free process at level $\lambda > 1$ if and only if there exists a probability measure Q λ-compatible to P which makes the discrete vector price process $\{Z_t\}$ a martingale.*

Proof: We prove the necessity part first. We begin by forming the dual problem to SP1. Attaching unrestricted-in-sign dual multiplier y_0 with the first constraint, multipliers $y_n, (n > 0)$ with the self-financing transaction constraints, and finally multipliers $w_n, (n \in \mathcal{N}_T)$ with the last set of constraints we form the Lagrangian function:

$$\begin{aligned} L(\Theta, X^+, X^-, y, w) &= \sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- \\ &\quad + y_0 Z_0 \cdot \theta_0 + \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} y_n Z_n \cdot (\theta_n - \theta_{\pi(n)}) + \sum_{n\in\mathcal{N}_T} w_n (Z_n \cdot \theta_n - x_n^+ + x_n^-) \end{aligned}$$

that we maximize over the variables Θ, X^+, and X^- separately. From these separate maximizations we obtain the following:

$$y_0 Z_0 = \sum_{n\in\mathcal{S}(0)} y_n Z_n \tag{2}$$

$$y_m Z_m = \sum_{n\in\mathcal{S}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 1 \leq t \leq T-1, \tag{3}$$

$$p_n \leq y_n \leq \lambda p_n, \ \forall n \in \mathcal{N}_T, \tag{4}$$

where we got rid of the dual variables w_n in the process by observing that maximizations over $\theta_n, (n \in \mathcal{N}_T)$ yield the equations

$$(w_n - y_n) Z_n = 0, \forall n \in \mathcal{N}_T,$$

and since the first component $Z_n^0 = 1$ for all states n, we have $y_n = w_n, (n \in \mathcal{N}_T)$. Therefore, we have obtained the dual problem that we refer to SD1 with an identically zero objective function and the constraints given by (2)–(3)–(4).

Now let us observe that problem SP1 is always feasible (the zero portfolio in all states is feasible) and if there is no SAGE opportunity, the optimal value is equal to zero. Therefore, by linear programming duality, the dual problem is also solvable (in fact, feasible since the dual is only a feasibility problem). Let us take any feasible solution $y_n, (n \in \mathcal{N})$ of the dual system given by (2)–(3)–(4). Since the first component, Z_n^0 is equal to 1 in each state n, we have that

$$y_m = \sum_{n \in \mathcal{S}(m)} y_n, \ \forall \, m \in \mathcal{N}_t, 1 \leq t \leq T-1. \tag{5}$$

Since $y_n \geq p_n$, it follows that y_n is a strictly positive process such that the sum of y_n over all states $n \in \mathcal{N}_t$ in each time period t sums to y_0. Now, define the process $q_n = y_n/y_0$, for each $n \in \mathcal{N}_T$. Obviously, this defines a probability measure Q over the leaf (terminal) nodes $n \in \mathcal{N}_T$. Furthermore, we can rewrite (3) with the newly defined weights q_n as

$$q_m Z_m = \sum_{n \in \mathcal{S}(m)} q_n Z_n, \ \forall \, m \in \mathcal{N}_t, 1 \leq t \leq T-1,$$

with $q_0 = 1$, and all $q_n > 0$. Therefore, by constructing the probability measure Q we have constructed an equivalent measure which makes the price process $\{Z_t\}$ a martingale according to Definition 1. By definition of the measure q_n, we have using the inequalities (4)

$$p_n \leq q_n y_0 \leq \lambda p_n, \forall n \in \mathcal{N}_T,$$

or equivalently,

$$p_n/q_n \leq y_0 \leq \lambda p_n/q_n, \forall n \in \mathcal{N}_T,$$

which implies that $q_n, n \in \mathcal{N}$ constitute a λ-compatible martingale measure. This concludes the necessity part.

Suppose Q is a λ-compatible martingale measure for the price process $\{Z_t\}$. Therefore, we have

$$q_m Z_m = \sum_{n \in \mathcal{S}(m)} q_n Z_n, \ \forall \, m \in \mathcal{N}_t, 1 \leq t \leq T-1,$$

with $q_0 = 1$, and all $q_n > 0$, while the condition $\max_{n \in \mathcal{N}_T} p_n/q_n \leq \lambda \min_{n \in \mathcal{N}_T} p_n/q_n$ holds. If the previous inequality holds as an equality, choose the right-hand (or, the left-hand) of the inequality as a factor y_0 and set $y_n = q_n y_0$ for all $n \in \Omega$. If the inequality is not tight, any value y_0 in the interval $[\max_{n \in \mathcal{N}_T} p_n/q_n, \lambda \min_{n \in \mathcal{N}_T} p_n/q_n]$ will do. It is easily verified that y_n, $n \in \mathcal{N}$ so defined satisfy the constraints of the dual problem SD1. Since the dual problem is feasible, the primal SP1 is bounded above (in fact, its optimal value is zero) and no SAGE opportunity exists in the system. ∎

As a first remark, we can immediately make a statement equivalent to Theorem 2: The price process (or the market) is SAGE-free at level λ if and only if there exists an equivalent measure Q to P such that:

$$\frac{\max_{n \in \mathcal{N}_T} p_n/q_n}{\min_{n \in \mathcal{N}_T} p_n/q_n} \leq \lambda \tag{6}$$

or, equivalently

$$\frac{\max_{n \in \mathcal{N}_T} q_n/p_n}{\min_{n \in \mathcal{N}_T} q_n/p_n} \leq \lambda \tag{7}$$

or,

$$\frac{\max_{\omega} \frac{dQ}{dP}(\omega)}{\min_{\omega} \frac{dQ}{dP}(\omega)} \leq \lambda \tag{8}$$

using the Radon-Nikodyn derivative, and that Q makes the price process a martingale. Clearly, posing the condition as such introduces a nonlinear system of inequalities, whereas our equivalent dual problem SD1 is a linear programming problem. After preparing this manuscript we noticed that a similar observation for single period problems was made in a technical note [16] although the language and notation of this reference is very different from ours.

As a second remark, we note that if we allow λ to tend to infinity we find ourselves in King's framework at which point Theorem 1 is valid. Therefore, this theorem is obtained as a special case of Theorem 2.

Example 1. Let us now consider a simple single-period numerical example. Let us assume for simplicity that the market consists of a riskless asset with zero growth rate, and of a stock. The stock price evolves according a trinomial tree as follows. Assume the riskless asset has price equal to one throughout. At time $t = 0$, the stock price is 10. Hence $Z_0 = (1\ 10)^T$. At the time $t = 1$, the stock price can take the values 20, 15, 7.5 with equal probability. Therefore, at node 1 one has $Z_1 = (1\ 20)^T$; at node 2 $Z_2 = (1\ 15)^T$ and finally at node 3 $Z_3 = (1\ 7.5)^T$. In other words, all β factors are equal to one. It is easy to see that the market described above is arbitrage free because we can show the existence of an equivalent martingale measure, e.g., $q_1 = q_2 = 1/8$ and $q_3 = 3/4$. Now, setting up and solving the problems SP1 and/or SD1, we observe that for all values of $\lambda \geq 6$, no SAGE opportunity exists in the market. However, for values of λ strictly between one and six, the primal problem SP1 is unbounded and the dual problem SD1 is infeasible. Therefore, SAGE opportunities exist.

As λ gets smaller, eventually the feasible set of the dual problem may reduce to a singleton, at which point an interesting pricing result is observed as we shall see in section 5. First, we investigate the problem of finding the smallest λ allowing no-sage opportunities in the next section.

4 Seeking out The Highest Possible λ in a SAGE Framework

We have assumed thus far that the parameter λ was decided by the agent (writer or buyer) before the solution of the stochastic linear programs of the previous section. However, once a SAGE opportunity is found at a certain level of λ it is legitimate to ask whether SAGE opportunities at higher levels of λ continue to exist. In fact, it is natural to wonder how far up one can push λ before SAGE opportunities cease to exist. Therefore, it is relevant, while seeking SAGE opportunities, to consider the following optimization problem LamP1:

$$\begin{array}{lrcl}
\sup & \lambda & & \\
\text{s.t.} & \displaystyle\sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- & > & 0 \\
& Z_0 \cdot \theta_0 & = & 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ & \geq & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- & \geq & 0,\ \forall\, n \in \mathcal{N}_T.
\end{array}$$

Notice that problem LamP1 is a non-convex optimization problem, and as such is potentially very hard. However, it can be posed in a form suitable for numerical processing. We can say that LamP1 is equivalent to the following problem referred to as LamPr

$$\begin{array}{lrcl}
\sup & \dfrac{\sum_{n\in\mathcal{N}_T} p_n x_n^+}{\sum_{n\in\mathcal{N}_T} p_n x_n^-} & & \\
\text{s.t.} & Z_0 \cdot \theta_0 & = & 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ & \geq & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- & \geq & 0,\ \forall\, n \in \mathcal{N}_T.
\end{array}$$

Notice that as a result of the homogeneity of the equalities and inequalities defining the constraints of problem LamPr, if Θ, X^+, X^- is feasible for LamPr, then so is $\kappa(\Theta, X^+, X^-)$ for any $\kappa > 0$, and the objective function value is constant along such rays. Under the assumption

Assumption 1 *The price process $\{Z_t\}$ is arbitrage-free, i.e., there does not exist feasible Θ, X^+, X^- with $\mathbb{E}^P[X^+] > 0$ and $\mathbb{E}^P[X^-] = 0$,*

we can now take one step further and say that problem LamPr is equivalent to problem LamPL:

$$\begin{array}{lrcl}
\max & \displaystyle\sum_{n\in\mathcal{N}_T} p_n x_n^+ & & \\
\text{s.t.} & \displaystyle\sum_{n\in\mathcal{N}_T} p_n x_n^- & = & 1 \\
& Z_0 \cdot \theta_0 & = & 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ & \geq & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- & \geq & 0,\ \forall\, n \in \mathcal{N}_T.
\end{array}$$

This equivalence can be established using the technique described on pp. 151 in [3] as follows. Let us take a solution Θ, X^+, X^- to LamPr, with $\xi^- = \sum_{n\in\mathcal{N}_T} p_n x_n^-$. It is easy to see that the point $\frac{1}{\xi^-}(\Theta, X^+, X^-)$ is feasible in LamPL with equal objective function value. For the converse, let $\Psi = (\Theta, X^+, X^-)$ be a feasible solution to LamPr, and let $\Xi = (\bar{\Theta}, \bar{X}^+, \bar{X}^-)$ be a feasible solution to LamPL. It is again immediate to see that $\Psi + t\Xi$ is feasible in LamPr for $t \geq 0$. Furthermore, we have

$$\lim_{t\to\infty} \frac{\mathbb{E}^P[X^+ + t\bar{X}^+]}{\mathbb{E}^P[X^- + t\bar{X}^-]} = \mathbb{E}^P[\bar{X}^+],$$

which implies that we can find feasible points in LamPr with objective values arbitrarily close to the objective function value at Ξ.

We can now construct the linear programming dual of LamPL using Lagrange duality technique which results in the dual linear program (HD1) in variables $y_n, (n \in \mathcal{N})$ and V:

$$\begin{aligned} \min \quad & V \\ \text{s.t.} \quad & y_m Z_m = \sum_{n\in\mathcal{S}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \leq t \leq T-1 \\ & p_n \leq y_n \leq V p_n, \ \forall n \in \mathcal{N}_T. \end{aligned}$$

Ley $Y(V)$ denote the set of $\{y_n\}$ that are feasible in the above problem for a given V. Notice that, for $V_1 < V_2$, one has $Y(V_1) \subseteq Y(V_2)$, assuming the respective sets to be non-empty. Hence, the optimal value of V is the minimum value such that the associated set $Y(V)$ is non-empty.

The dual can also be re-written as (HD2):

$$\begin{aligned} \min \quad & \max_{n\in\mathcal{N}_T} \frac{y_n}{p_n} \\ \text{s.t.} \quad & y_m Z_m = \sum_{n\in\mathcal{S}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \leq t \leq T-1 \\ & p_n \leq y_n, \ \forall n \in \mathcal{N}_T. \end{aligned}$$

Let Y denote the set of feasible solutions to the above problem. We summarize our findings in the proposition below.

Proposition 1 *Under Assumption 1 we have*

1. *Problem LamP1 is equivalent to problem LamPL.*
2. *When optimal solutions exist, for any optimal solution $\Theta^*, (X^+)^*, (X^-)^*, \lambda^*$ of LamP1, we have that $\frac{1}{\mathbb{E}^P[(X^-)^*]}(\Theta^*, (X^+)^*, (X^-)^*)$ is optimal for LamPL.*
3. *When optimal solutions exist, for any optimal solution $\Theta^*, (X^+)^*, (X^-)^*$ of LamPL and any $\kappa > 0$, we have that $\kappa(\Theta^*, (X^+)^*, (X^-)^*), \frac{\mathbb{E}^P[(X^+)^*]}{\mathbb{E}^P[(X^-)^*]}$ is optimal for LamP1.*
4. *When optimal solutions exist the largest possible value λ^* of λ is equal to $\min_{y\in Y} \max_{n\in\mathcal{N}_T} \frac{y_n}{p_n}$.*

The last item of the above proposition is essentially the duality result of Bernardo and Ledoit (c.f. Theorem 1 on page 151 of [1]) which they prove for single period investments but using an infinite-state setup.

By way of illustration, setting up and solving the problem LamPL for the trinomial numerical example of the previous section, one obtains the largest value of λ as six, as the optimal value of the problem LamPL. This is the smallest value of λ that does not allow a SAGE opportunity. Put in other words, it is the supremum of all values of λ allowing a SAGE.

5 Financing of Contingent Claims and Sufficiently Attractive Expected Gains: Positions of Writers and Buyers

Now, let us take the viewpoint of a writer of contingent claim F which is generating pay-offs $F_n, (n > 0)$ to the holder (liabilities of the writer), depending on the states n of the market (hence the adjective contingent). The following is a legitimate question on the part of the writer: what is the minimum initial investment needed to replicate the pay-outs F_n using securities available in the market with no risk of positive expected terminal wealth falling short of λ times the expected negative terminal wealth? King [15] posed a similar question in the context of no-arbitrage pricing, hence for preventing the risk of terminal positions being negative at any state of nature. Here, obviously we are working with an enlarged feasible set of replicating portfolios, if not empty.

Let us now pose the problem of financing of the writer who opts for the SAGE viewpoint at level λ rather than the classical arbitrage viewpoint. The writer is facing the stochastic linear programming problem WP1

$$\begin{array}{lrcl} \min & Z_0 \cdot \theta_0 & & \\ \text{s.t.} & Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0, \ \forall\, n \in \mathcal{N}_T, \\ & \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- & \geq & 0 \\ & x_n^+ & \geq & 0, \ \forall\, n \in \mathcal{N}_T, \\ & x_n^- & \geq & 0, \ \forall\, n \in \mathcal{N}_T, \end{array}$$

as opposed to King's financing problem

$$\begin{array}{lrcl} \min & Z_0 \cdot \theta_0 & & \\ \text{s.t.} & Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n & \geq & 0, \ \forall\, n \in \mathcal{N}_T. \end{array}$$

Let us assume that a price of F_0 is attached to a contingent claim F. The following definition is useful.

Definition 4 *A contingent claim F with price F_0 is said to be λ-attainable if there exists $\Theta = (\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|})$ satisfying:*

$$Z_0 \cdot \theta_0 \leq \beta_0 F_0,$$

$$Z_n \cdot (\theta_n - \theta_{\pi(n)}) = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1$$

and

$$\mathbb{E}^P[X^+] - \lambda\mathbb{E}^P[X^-] = 0.$$

Proposition 2 *At a fixed level* $\lambda > 1$, *assume the discrete vector price process* $\{Z_t\}$ *is SAGE-free. Then the minimum initial investment* W_0 *required to hedge the claim with no risk of expected positive terminal wealth falling short of* λ *times the expected negative terminal wealth satisfies*

$$W_0 = \frac{1}{y_0^* \beta_0} \max_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n$$

where $Y(\lambda)$ *is the set of all* $y \in \mathbb{R}^{|\mathcal{N}|}$ *satisfying the conditions (2)– (3)–(4), i.e., the feasible set of SD1, and* y_0^* *is the first component of any maximizing vector* y^*.

Proof: Let us begin by forming the linear programming dual of problem SP2. Forming the Lagrangian function after attaching multipliers $v_n, (n > 0)$, $w_n, (n \in \mathcal{N}_T)$ (all unrestricted-in-sign) and $V \geq 0$ we obtain

$$\begin{aligned} L(\Theta, X^+, X^-, v, w, V) &= Z_0 \cdot \theta_0 + V(\lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- - \sum_{n \in \mathcal{N}_T} p_n x_n^+) + \\ & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} v_n \left(Z_n \cdot (\theta_n - \theta_{\pi(n)}) + \beta_N F_n\right) + \sum_{n \in \mathcal{N}_T} w_n (Z_n \cdot \theta_n - x_n^+ + x_n^-) \end{aligned}$$

that we maximize over the variables Θ, X^+, and X^- separately again. This results in the dual problem WD2.1

$$\begin{array}{lrcl} \max & \displaystyle\sum_{n>0} v_n \beta_n F_n & & \\ \text{s.t.} & Z_0 &=& \displaystyle\sum_{n \in \mathcal{S}(0)} v_n Z_n \\ & v_m Z_m &=& \displaystyle\sum_{n \in \mathcal{S}(m)} v_n Z_n, \ \forall\, m \in \mathcal{N}_t, 1 \leq t \leq T-1 \\ & V p_n \leq v_n &\leq& V \lambda p_n, \ \forall n \in \mathcal{N}_T, \\ & V &\geq& 0. \end{array}$$

We observe that no feasible solution to WD2.1 could have a V-component equal to zero as this would lead to infeasibility in the v-component. Therefore, it is easy to see that the dual is equivalent to the problem (that we refer to as WD2.2) up to a multiplicative constant:

$$\begin{array}{lrcl} \max & \displaystyle\sum_{n>0} y_n \beta_n F_n & & \\ \text{s.t.} & y_m Z_m &=& \displaystyle\sum_{n \in \mathcal{S}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \leq t \leq T-1 \\ & p_n \leq y_n &\leq& \lambda p_n, \ \forall n \in \mathcal{N}_T. \end{array}$$

This multiplicative constant is given by $1/y_0^*$ for any optimal solution y^* of WD2.2. However, the feasible set of the previous problem is identical to the feasible set $Y(\lambda)$ of the dual SD1 in Proposition

1. Therefore, if the price process $\{Z_t\}$ is SAGE-free, then there exists a feasible solution to the dual SD1, and hence, a feasible solution to the dual problem WD2.2. Since WD2.2 is feasible and bounded above, the primal problem WP1 is solvable by linear programming duality theory. Hence, the result follows. ∎

Notice that in the previous proof we obtained two equivalent expressions for the dual problem of WP1, namely the dual problem in the statement of the Proposition 3 or WD2.2, and the problem WD2.1. For future reference, we refer to the feasible set of WD2.1 as $Q(\lambda)$, and to its projection on the set of v's as $\bar{Q}(\lambda)$. Since we observed that no optimal (in fact, feasible) solution to WD2.1 could have a V-component equal to zero as this would lead to infeasibility in the v-component, by the complementary slackness property of optimal solutions to the primal and the dual problems in linear programming, we should have in all optimal solutions (Θ, X^+, X^-) to the primal:

$$\mathbb{E}^P[X^+] - \lambda \mathbb{E}^P[X^-] = 0.$$

We immediately have the following.

Corollary 5.1 *At fixed level $\lambda > 1$, assume the discrete vector price process $\{Z_t\}$ is SAGE-free. Then, contingent claim F priced at F_0 is λ-attainable if and only if*

$$\beta_0 y_0^* F_0 \geq \max_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n$$

where y_0^* is the first component of any maximizing vector y^*.

In the light of the above, the minimum acceptable price to the writer of the contingent claim F is given by the expression

$$F_0^w = \frac{1}{y_0^* \beta_0} \max_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n. \tag{9}$$

Let us now look at the problem from the viewpoint of a potential buyer. The buyer's problem is to decide the maximum price he/she should pay to acquire the claim, with no risk of expected positive terminal wealth falling short of λ times the expected negative terminal wealth. This translates into the problem

$$\begin{array}{lrcl}
\max & -Z_0 \cdot \theta_0 & & \\
\text{s.t.} & Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & \beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
 & Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0, \ \forall\, n \in \mathcal{N}_T, \\
 & \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- & \geq & 0 \\
 & x_n^+ & \geq & 0, \ \forall\, n \in \mathcal{N}_T, \\
 & x_n^- & \geq & 0, \ \forall\, n \in \mathcal{N}_T.
\end{array}$$

The interpretation of this problem is the following: find the maximum amount needed for acquiring a portfolio replicating the proceeds from the contingent claim without the risk of expected negative wealth magnified by a factor λ exceeding the expected positive terminal wealth. By repeating the analysis done for the writer (that we do not reproduce here), we can assert that the maximum acceptable price F_0^b to the buyer in our framework is given by the following, provided that the price process $\{Z_t\}$ is SAGE-free at level λ:

$$F_0^b = \frac{1}{y_0^* \beta_0} \min_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n. \tag{10}$$

Therefore, for fixed $\lambda > 1$ and P, we can conclude that the writer's minimum acceptable price and the buyer's maximum acceptable price in a SAGE-free market constitute a price interval given as

$$[\frac{1}{y_0^* \beta_0} \min_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n; \frac{1}{y_0^* \beta_0} \max_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n].$$

We could equally express this interval as

$$[\frac{1}{\beta_0} \min_{v, V \in Q(\lambda)} \mathbb{E}^v [\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{v, V \in Q(\lambda)} \mathbb{E}^v [\sum_{t=1}^{T} \beta_t F_t]].$$

This is the interval of prices which do not induce either the buyer or writer to engage in buying or selling the contingent claim. They can also be thought of as bounds on the price of the contingent claim. Let us recall that the no-arbitrage pricing interval obtained by King [15] corresponds to

$$[\frac{1}{\beta_0} \min_{q \in \bar{Q}} \mathbb{E}^q [\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{q \in \bar{Q}} \mathbb{E}^q [\sum_{t=1}^{T} \beta_t F_t]];$$

where $\bar{Q}$ is the set of $q \in \mathbb{R}^{|\mathcal{N}|}$ satisfying

$$Z_0 = \sum_{n \in \mathcal{S}(0)} q_n Z_n$$

$$q_m Z_m = \sum_{n \in \mathcal{S}(m)} q_n Z_n, \ \forall \, m \in \mathcal{N}_t, 1 \leq t \leq T-1$$

and

$$q_n \geq 0 \ \forall n \in \mathcal{N}_T.$$

Clearly, for fixed λ we have the inclusion $\bar{Q}(\lambda) \subset \bar{Q}$ using the positivity of V. Hence, the pricing interval obtained above is a smaller interval in width in comparison to the arbitrage-free pricing interval of [15]. Notice that the two intervals will become indistinguishable as λ tends to infinity. The more interesting question is the behavior of the interval as λ is decreased. To examine this issue, consider solving the smallest-no-SAGE λ-seeking problem LamPL or its dual HD1 (or HD2). If one solves the dual problem HD1 to obtain as optimal solutions V^*, y^*, and if this solution is the unique feasible solution to the linear program HD1, i.e., if the set of equations and inequalities defining the constraints of HD1 for the fixed value of V^* admit a unique solution vector y^*, then this immediately implies that the no-sage pricing bounds at level $\lambda = V^*$, i.e., the bounds $\frac{1}{y_0^* \beta_0} \min_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n$, $\frac{1}{y_0^* \beta_0} \max_{y \in Y(\lambda)} \sum_{n>0} y_n \beta_n F_n$ coincide since both problems possess the common single feasible point y^*.

While the property of uniqueness of optimal solution V^*, y^* to HD1 appeared to hold in all our computational experience, we do not know whether the property always holds. It appears to be a non-trivial problem to ascertain or refute the truthfulness of this claim. On the other hand, our practical experience indicates that at the smallest possible no-SAGE allowing λ value, the no-SAGE bounds converge to a single point.

Example 2. Consider the same simple market model of Example 1 in Section 3. We assume a contingent claim on the stock, of the European Call type with a strike price equal to 9 is available. Therefore, we have the following pay-off structure: $F_1 = 11, F_2 = 6, F_3 = 0$, corresponding to nodes 1, 2 and 3, respectively. Computing the no-arbitrage bounds using linear programming, one obtains the interval of prices $[2.0; 2.2]$ corresponding to the buyer and to the writer's problems respectively. For $\lambda = 8$, the SAGE-free price interval is $[2.09; 2.14]$. For $\lambda = 7$, the interval becomes $[2.10; 2.13]$. Finally, for $\lambda = 6$, which is the smallest allowable value for λ below which the above derivations lose their validity, the interval shrinks to a single value of 2.125, since both the buyer and the writer problems return the same optimal value. Therefore, for two investors that are ready to accept an expected gain prospect that is at least six times as large as an expected loss prospect, it is possible to agree on a common price for the contingent claim in question. In this particular example, the problem HD1 for $\lambda = 6$ which is the optimal value for λ, possesses a single feasible point $y = (2.66, 0.33, 0.33, 2)^T$. Dividing the components by 2.66 which is the component y_0, we obtain the unique equivalent martingale measure $(1, 1/8, 1/8, 3/4)^T$ (which is also λ-compatible) leading to the unique price of the contingent claim.

Interestingly, the hedging policies of the buyer and the writer at level $\lambda = 6$ need not be identical. For the writer an optimal hedging policy is to short 6.75 units of riskless asset at $t = 0$ and buy 0.887 units of the stock. If node 1 were to be reached, the hedging policy dictates to liquidate the position in both the bond and the stock. In case of node 2, the position in the stock is zeroed out, and a position of 0.562 units in the bond is taken. Finally in node 3, the position in the stock is zeroed out, but a short position of 0.094 units remains in the riskless asset. For the buyer an optimal hedging policy is to buy 5.625 units of riskless asset at $t = 0$ and short 0.775 units of the stock. At time $t = 1$ if node 1 were to be reached, the hedging policy dictates to pass to a position of 1.125 units in the bond, and to a zero position in the stock. In case of node 2, all positions are zeroed out. Finally in node 3, the position in the stock is zeroed out, but a short position of 0.187 units remains in the riskless asset.

Example 3. Let us now consider a two-period version of the previous example. The market is again described through a trinomial structure. Let the asset price be as in Example 1 and 2 for time $t = 1$. At time $t = 2$, from node 1 at which the price is 20, the price can evolve to 22, 21 and 19 with equal probability, thereby giving the asset price values at nodes 4, 5 and 6. From node 2 at which the price takes value equal to 15, the price can go to 17 or 14 or 13 with equal probability, resulting in the asset price values at nodes 7, 8 and 9. Finally, from node 3, we have as children nodes the node 10, node 11 and node 12, with equally likely asset price realizations equal to 9, 8 and 7, respectively. Therefore, the trinomial tree contains 9 paths, each with a probability equal to $1/9$. The riskless asset is assumed to have value one throughout. It can be verified that this market is arbitrage free.

Solving for the supremum of λ values allowing a SAGE, we obtain 14.5.

Node	B	S
0	-4.056	0.503
1	-14	1
2	7.13	-0.243
3	-4.563	0.57
8	3.729	
9	3.972	
10	0.57	
12	-0.57	

Table 1: The writer's optimal hedge policy for $\lambda = 14.5$.

Node	B	S
0	-0.915	-0.006
1	-80.465	3.972
2	14	-1
3	-15.324	1.915
4	14.915	
5	9.944	
9	1	
10	1.915	
12	-1.915	

Table 2: The buyer's optimal hedge policy for $\lambda = 14.5$.

Now, let us assume we have a European Call option F on the stock with strike price equal to 14, resulting in pay-off values $F_4 = 8$, $F_5 = 7$, $F_6 = 5$ and $F_7 = 3$ where the index corresponds to the node number in the tree (all other values F_n are equal to zero). The no-arbitrage bounds yield the interval $[0.33, 1.2]$ for this contingent claim. The no-SAGE intervals go as follows: for $\lambda = 17$ one has $[0.86; 1.00]$, for $\lambda = 16$, $[0.9; 0.99]$, for $\lambda = 15$ $[0.94; 0.98]$. For the limiting value of 14.5 the bounds again collapse to a single price of 0.9718 attained at the same λ-equivalent martingale measure $q_4 = q_5 = 0.028$, $q_6 = 0.085$, $q_7 = 0.042$, $q_8 = q_9 = q_{10} = 0.028$, $q_{11} = 0.324$ and $q_{12} = 0.408$.

Two tables, Table 1 and Table 2, summarize the optimal hedge policies of the writer and the buyer, respectively, when the single price is reached. We only report the results for nodes where non-zero portfolio positions are held. The symbols B and S stand for the riskless asset and the stock, respectively. Again, the hedge policies are quite different, but result in an identical price.

Notice that the analysis of the writer's and buyer's hedging problems can be also be done using a simple utility function for modeling risk aversion and the conjugate duality framework of convex optimization [17]. The utility function corresponding to no-arbitrage is given as

$$u_w(v) = v - I_{v\geq 0}(v)$$

where $I_{v\geq 0}$ is the indicator function of convex analysis which equals zero if $v \geq 0$, and $+\infty$ otherwise. Our problems involving the gain-loss objective function (and/or constraint) could alternatively be modeled using the equally simple piecewise-linear utility function

$$u(v) = \begin{cases} v & \text{if } v \geq 0 \\ \lambda v & \text{if } v < 0. \end{cases}$$

Then, all our results could be obtained using the concave conjugate function u^* given by

$$u^*(y) = \inf_v (yv - u(v))$$

which is finite in our case (in fact, zero) provided that $1 \leq y \leq \lambda$, which are exactly the constraints showing up in our dual problems where the argument of the u^* function is precisely y_n/p_n.

In closing this section we point out that Bernardo and Ledoit's gain-loss ratio results that were obtained in a single-period, non-linear optimization framework are very similar to the approach described above. We showed that similar results can be obtained in a multi-period (finite probability), linear optimization setting, which is simpler yet much more intuitive.

6 Proportional Transaction Costs

The problem of hedging and pricing contingent claims in presence of transaction costs was investigated in e.g. [8, 11, 12]. In [8], it was assumed that the cost of trading a stock (excluding the numéraire) is proportional to the price. An investor who buys one share of stock j when the (discouned with respect to the numéraire) stock price is Z_t^j pays $Z_t^j(1+\eta)$ whereas upon establishing a short position the investor gets $Z_t^j(1-\eta)$. Let us now denote the components of Z_t corresponding to the indices from 1 to J, as the vector $\bar{Z}_t$. Similarly, we refer to the components of Z_n corresponding to the indices from 1 to J, as the vector $\bar{Z}_n$, and as $\bar{\theta}_n$ to the portfolio positions corresponding to all these stocks excluding the numéraire, for node n of the scenario tree. Then, the arbitrage problem becomes the following:

$$\begin{array}{lrcl} \max & \displaystyle\sum_{n\in\mathcal{N}_T} p_n Z_n\cdot\theta_n & & \\ \text{s.t.} & \theta_0^0 + \bar{Z}_0\cdot\bar{\theta}_0 + \eta\bar{Z}_0\cdot|\bar{\theta}_0| & = & 0 \\ & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n\cdot(\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \bar{Z}_n\cdot|\bar{\theta}_n - \bar{\theta}_{\pi(n)}| & = & 0,\ \forall\, n\in\mathcal{N}_t, t\geq 1 \\ & Z_n\cdot\theta_n & \geq & 0,\ \forall\, n\in\mathcal{N}_T, \end{array}$$

where the absolute value operator is applied componentwise.

Using a standard trick to transform the above problem into a linear program, and linear programming duality, one can prove the following result.

Theorem 3 *The discrete state stochastic vector process* $\{Z_t\}$ *is an-arbitrage free market price process if and only if there is at least one probability measure* Q *equivalent to* P *under which the process* $\{Z_t\}$ *fulfills*

$$(1-\alpha)\bar{Z}_t \leq \mathbb{E}^Q[\bar{Z}_{t+1}|\mathcal{N}_t] \leq (1+\alpha)\bar{Z}_t, \ \ \forall t \leq T-1. \tag{11}$$

The proof is similar in essence to that of Theorem 2 (or that of Theorem 1 in [15]) hence, is omitted. Note that for $\eta = 0$ one recovers Theorem 1.

The SAGE-seeking investor (at a fixed λ) is interested in solving the problem:

$$\begin{array}{lrcl}
\max & \displaystyle\sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- & & \\
\text{s.t.} & \theta_0^0 + \bar{Z}_0\cdot\bar{\theta}_0 + \eta\bar{Z}_0\cdot|\bar{\theta}_0| & = & 0 \\
& \theta_n^0 - \theta_{\pi(n)}^0 + Z_n\cdot(\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + Z_n\cdot|\bar{\theta}_n - \bar{\theta}_{\pi(n)}| & = & 0, \ \forall\, n\in\mathcal{N}_t, t\geq 1 \\
& Z_n\cdot\theta_n - x_n^+ + x_n^- & = & 0, \ \forall\, n\in\mathcal{N}_T, \\
& x_n^+ & \geq & 0, \ \forall\, n\in\mathcal{N}_T, \\
& x_n^- & \geq & 0, \ \forall\, n\in\mathcal{N}_T.
\end{array}$$

The counterpart of Theorem 2 in this case becomes the following.

Theorem 4 *The discrete state stochastic vector process* $\{Z_t\}$ *is a SAGE-free market price process at level* λ *if and only if there is at least one probability measure* Q λ*-compatible to* P *under which the process* $\{Z_t\}$ *fulfills (11).*

For $\eta = 0$ one recovers Theorem 2.

Now, the no-arbitrage price bounds of the previous section are computed by solving

$$\begin{array}{lrcl}
\min & \theta_0^0 + \bar{Z}_0\cdot\bar{\theta}_0 + \eta\bar{Z}_0\cdot|\bar{\theta}_0| & & \\
\text{s.t.} & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n\cdot(\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \bar{Z}_n\cdot|\bar{\theta}_n - \bar{\theta}_{\pi(n)}| & = & -\beta_n F_n, \ \forall\, n\in\mathcal{N}_t, t\geq 1 \\
& Z_n\cdot\theta_n & \geq & 0, \ \forall\, n\in\mathcal{N}_T.
\end{array}$$

for the writer, and

$$\begin{array}{lrcl}
\max & -\theta_0^0 - \bar{Z}_0\cdot\bar{\theta}_0 - \eta\bar{Z}_0\cdot|\bar{\theta}_0| & & \\
\text{s.t.} & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n\cdot(\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \bar{Z}_n\cdot|\bar{\theta}_n - \bar{\theta}_{\pi(n)}| & = & \beta_n F_n, \ \forall\, n\in\mathcal{N}_t, t\geq 1 \\
& Z_n\cdot\theta_n & \geq & 0, \ \forall\, n\in\mathcal{N}_T,
\end{array}$$

for the buyer. These bounds are also obtained using the dual expressions:

$$[\frac{1}{\beta_0}\min_{q\in\tilde{Q}}\mathbb{E}^Q[\sum_{t=1}^T \beta_t F_t]; \frac{1}{\beta_0}\max_{q\in\tilde{Q}}\mathbb{E}^Q[\sum_{t=1}^T \beta_t F_t]].$$

where $\tilde{Q}$ is the (closure of) set of equivalent measures to P such that the process $\{Z_t\}$ satisfies condition (11).

Now, let us consider the no-SAGE bounds obtained from the perspective of the buyer and the writer. Going through the usual problems in the hedging space:

$$\begin{array}{ll} \min & \theta_0^0 + \bar{Z}_0 \cdot \bar{\theta}_0 + \eta \bar{Z}_0 \cdot |\bar{\theta}_0| \\ \text{s.t.} & \begin{aligned} \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \bar{Z}_n \cdot |\bar{\theta}_n - \bar{\theta}_{\pi(n)}| &= -\beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\ Z_n \cdot \theta_n - x_n^+ + x_n^- &= 0, \; \forall\, n \in \mathcal{N}_T, \\ x_n^+ &\geq 0, \; \forall\, n \in \mathcal{N}_T, \\ x_n^- &\geq 0, \; \forall\, n \in \mathcal{N}_T \end{aligned} \end{array}$$

for the writer, and

$$\begin{array}{ll} \max & -\theta_0^0 - \bar{Z}_0 \cdot \bar{\theta}_0 - \eta \bar{Z}_0 \cdot |\bar{\theta}_0| \\ \text{s.t.} & \begin{aligned} \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \bar{Z}_n \cdot |\bar{\theta}_n - \bar{\theta}_{\pi(n)}| &= \beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\ Z_n \cdot \theta_n - x_n^+ + x_n^- &= 0, \; \forall\, n \in \mathcal{N}_T, \\ x_n^+ &\geq 0, \; \forall\, n \in \mathcal{N}_T, \\ x_n^- &\geq 0, \; \forall\, n \in \mathcal{N}_T \end{aligned} \end{array}$$

for the buyer, one obtains by using duality that the corresponding interval of no-SAGE at level λ is

$$[\frac{1}{\beta_0} \min_{q \in \tilde{Q}(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{q \in \tilde{Q}(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t]]$$

where $\tilde{Q}(\lambda)$ is the set of λ-compatible measures to P such that the process $\{Z_t\}$ satisfies (11). Obviously, the no-SAGE bounds are tighter compared to the no-arbitrage bounds.

Example 4. Considering the same problem as in example 2 with $\eta = 0.1$, the supremum of the values of λ allowing a SAGE opportunity is computed to 3.715 (notice the drop from 6 in the case of no transaction costs). The no-arbitrage interval for the contingent claim is found to be $[1.2; 3.08]$. At $\lambda = 4$, the no-SAGE interval is $[2.83; 2.98]$. At $\lambda = 3.715$ which is the limiting value, the common bound is equal to 2.97. The unique measure leading to this common price is given as $q_1 = q_2 = 0.175$ and $q_3 = 0.65$.

7 Losses as Conditional Value-at-Risk

In the previous sections, we modeled the risk component of the objective function using the expected value of negative terminal wealth positions. It is also possible to develop the concept of sufficiently attractive expected gains using the conditional Value-at-Risk measure (CVaR) of Rockafellar and Uryasev [18]. CVaR (also referred to as Average Value-at-Risk) was developed as a convenient alternative to Value-at-Risk (VaR) which is the maximum predicted loss associated with an investment decision with a specified confidence probability level α (e.g., 0.95). However, the VaR is usually criticized for failing to consider the magnitude of potential losses beyond itself. It also presents difficulties at the computational level in that the VaR associated with a portfolio is usually a non-convex function of the portfolio holdings. In contrast, CVaR aims to compute the expected loss given that loss exceeded VaR at a given percentile. Under the assumption of a discrete probability measure governing the loss process (as is the case in this paper), one can compute CVaR minimizing portfolios using linear programming as shown by Rockafellar and Uryasev [18]. It is precisely this feature of CVaR that we exploit in the present section. We list the main results without repeating all the details that are similar to the previous sections.

In this particular case, our point of departure is the stochastic linear program in the variables Θ, X^+, X^-, γ at a fixed λ and confidence level α (e.g., 0.95):

$$\begin{aligned} \max \quad & \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda f_\alpha(X^-, \gamma) \\ \text{s.t.} \quad & Z_0 \cdot \theta_0 \;=\; 0 \\ & Z_n \cdot (\theta_n - \theta_{\pi(n)}) \;=\; 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n - x_n^+ + x_n^- \;=\; 0,\ \forall\, n \in \mathcal{N}_T, \\ & x_n^+ \;\geq\; 0,\ \forall\, n \in \mathcal{N}_T, \\ & x_n^- \;\geq\; 0,\ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

where

$$f_\alpha(X^-, \gamma) := \gamma + \frac{1}{1-\alpha} \sum_{n \in \mathcal{N}_T} p_n \max(0, x_n^- - \gamma).$$

We will say that the price process $\{Z_t\}$ does not admit a CVaR-SAGE at level λ and confidence α if the optimal value of the above problem is equal to zero. It is well-known that minimizing the function f_α jointly over γ and portfolio holdings (in our case Θ) yields the CVaR at α-confidence level, while the optimal value of γ gives the Value-at-Risk (VaR) at the same percentile α. Introducing auxiliary variables u_n we can pose the problem as the following:

$$\begin{aligned} \max \quad & \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda(\gamma + \frac{1}{1-\alpha} \sum_{n \in \mathcal{N}_T} p_n u_n) \\ \text{s.t.} \quad & Z_0 \cdot \theta_0 \;=\; 0 \\ & Z_n \cdot (\theta_n - \theta_{\pi(n)}) \;=\; 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n - x_n^+ + x_n^- \;=\; 0,\ \forall\, n \in \mathcal{N}_T, \\ & x_n^+ \;\geq\; 0,\ \forall\, n \in \mathcal{N}_T, \\ & x_n^- \;\geq\; 0,\ \forall\, n \in \mathcal{N}_T, \\ & u_n \;\geq\; 0,\ \forall\, n \in \mathcal{N}_T, \\ & u_n \;\geq\; x_n^- - \gamma,\ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

Constructing the dual of the above linear program we obtain the feasibility problem in the variables $y_n, \forall n \geq 0$ and $w_n, \forall n \in \mathcal{N}_T$

$$y_m Z_m = \sum_{n \in \mathcal{S}(m)} y_n Z_n,\ \forall\, m \in \mathcal{N}_t, 0 \leq t \leq T-1, \tag{12}$$

$$y_0 = \lambda, \tag{13}$$

$$p_n \le y_n \le \frac{\lambda}{1-\alpha} p_n,\ \forall n \in \mathcal{N}_T, \tag{14}$$

Passing to martingale measures, the dual problem could also be equivalently written as the feasibility problem of the system

$$q_m Z_m = \sum_{n \in \mathcal{S}(m)} q_n Z_n,\ \forall\, m \in \mathcal{N}_t, 0 \le t \le T-1, \tag{15}$$

$$q_0 = 1, \tag{16}$$

$$\frac{1}{\lambda} p_n \le q_n \le \frac{1}{1-\alpha} p_n,\ \forall n \in \mathcal{N}_T. \tag{17}$$

This latter system gives us the means to extend a martingale measure equivalent to P in such a way to obtain a feasible point $y_n, \forall n \ge 0$ for the system (12)–(14). To this end we define a (α, λ)-compatible measure.

Definition 5 *Given $\lambda > 1$ and $\alpha \in [0,1]$ and a discrete probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ is (α, λ)-compatible to a (discrete) probability measure $P = \{p_n\}_{n \in \mathcal{N}_T}$ if it is equivalent to P (Definition 2) and satisfies*

$$\frac{1}{\lambda} \max_{n \in \mathcal{N}_T} p_n / q_n \le 1 \le \frac{1}{1-\alpha} \min_{n \in \mathcal{N}_T} p_n / q_n.$$

We can immediately state the counterpart result of Theorem 2. The spirit of the proof is similar to that of Theorem 2, mutatis mutandis, hence the details are left as an exercise.

Theorem 5 *The discrete state stochastic vector process $\{Z_t\}$ is a CVaR-SAGE-free market price process at a fixed level λ and confidence level α if and only if there is at least one probability measure Q (α, λ)-compatible to P under which $\{Z_t\}$ is a martingale.*

Notice that for $\alpha = 0$ and $\lambda = \infty$ one recovers Theorem 1. For $\alpha = 0$, Theorem 2 is obtained.

As in section 4, one can solve the following optimization problem to find the value of the supremum of λ values allowing a CVaR-SAGE at confidence α:

$$\begin{array}{lrcl}
\max & \displaystyle\sum_{n \in \mathcal{N}_T} p_n x_n^+ & & \\
\text{s.t.} & \displaystyle\gamma + \frac{1}{1-\alpha} \sum_{n \in \mathcal{N}_T} p_n u_n & = & 1 \\
& Z_0 \cdot \theta_0 & = & 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0,\ \forall\, n \in \mathcal{N}_t, t \ge 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- & = & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ & \ge & 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- & \ge & 0,\ \forall\, n \in \mathcal{N}_T, \\
& u_n & \ge & 0,\ \forall\, n \in \mathcal{N}_T, \\
& u_n & \ge & x_n^- - \gamma,\ \forall\, n \in \mathcal{N}_T.
\end{array}$$

We notice that as α is made zero, we recover the problem LamPL, since we can zero out γ (since the Value-at-risk at zero confidence level is zero), and get rid of the last two sets of constraints involving the u-variables (and replace the u_n with x_n^- in the first constraint) in this case. In fact, as α is decreased to zero gradually, we observed in our computational experience that the supremum of λ values converges to the supremum of λ values in the SAGE case of the previous sections. This is to be expected since as α becomes smaller, the effect of CVaR on the optimization becomes less pronounced, and tends to measure risk by the expected value of negative positions.

Similarly to section 5, the dual problems to the writer's hedging problem and the buyer's hedging problem for a contingent claim F with price F_0 and pay-out scheme $F_n, n > 0$ lead to lower and upper price bounds (after division by $\beta_0 y_0^*$ as in the previous section), and are given as the following problems:

$$
\begin{array}{lrcl}
\max & \sum_{n>0} y_n \beta_n F_n & & \\
\text{s.t.} & y_m Z_m & = & \sum_{n \in \mathcal{S}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \le t \le T-1 \\
& y_0 & = & \lambda, \\
& p_n \le y_n & \le & \frac{\lambda}{1-\alpha} p_n, \ \forall n \in \mathcal{N}_T,
\end{array}
$$

$$
\begin{array}{lrcl}
\min & \sum_{n>0} y_n \beta_n F_n & & \\
\text{s.t.} & y_m Z_m & = & \sum_{n \in \mathcal{S}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \le t \le T-1 \\
& y_0 & = & \lambda, \\
& p_n \le y_n & \le & \frac{\lambda}{1-\alpha} p_n, \ \forall n \in \mathcal{N}_T,
\end{array}
$$

Under the assumption of existence of a martingale measure (α, λ)-compatible to P, these two bounds are computable, and again yield a smaller pricing interval compared to the no-arbitrage pricing intervals. Stated using probability measures directly, these bounds are equivalently given as:

$$
\begin{array}{lrcl}
\max & \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t] & & \\
\text{s.t.} & Z_0 & = & \sum_{n \in \mathcal{S}(0)} q_n Z_n \\
& q_m Z_m & = & \sum_{n \in \mathcal{S}(m)} q_n Z_n, \ \forall\, m \in \mathcal{N}_t, 1 \le t \le T-1 \\
& \frac{p_n}{\lambda} \le q_n & \le & \frac{p_n}{1-\alpha}, \ \forall n \in \mathcal{N}_T,
\end{array}
$$

for the writer, and

$$\begin{array}{lrcl}\min & \mathbb{E}^Q[\sum_{t=1}^{T}\beta_t F_t] & & \\ \text{s.t.} & Z_0 & = & \sum_{n\in\mathcal{S}(0)} q_n Z_n \\ & q_m Z_m & = & \sum_{n\in\mathcal{S}(m)} q_n Z_n,\ \forall\, m\in\mathcal{N}_t, 1\le t\le T-1 \\ & \frac{p_n}{\lambda}\le q_n & \le & \frac{p_n}{1-\alpha},\ \forall n\in\mathcal{N}_T,\end{array}$$

for the buyer, respectively. The transition between y's and q's is through the formula $y_n = \lambda q_n$, for all $n \geq 0$.

Example 5. For the same contingent claim as in Example 2, and in an identical trinomial market setting, we obtain $\lambda = 8/3$ as the supremum of values of λ allowing a CVaR-SAGE at confidence level 0.95. This value is smaller than the limiting value of 6 for the case of losses measured by their expected values. For $\lambda = 5$ the buyer-writer no CVaR-SAGE interval is $[2.06; 2.16]$. For $\lambda = 4$, it is $[2.083; 2.15]$, and $\lambda = 3$ $[2.111; 2.133]$. The bounds converge at $\lambda = 8/3$ to the unique value 2.125. Therefore, two investors that are ready to accept an expected gain prospect that is at least $8/3$ times as large as an expected loss prospect, and valuing losses using CVaR measure at 0.95 confidence level, it is possible to agree on a common price equal to 2.125 for the contingent claim in question.

Example 6. For the two-period problem of Example 3, the equilibrium price of the contingent claim which is equal to 0.9718 is reached by the buyer and the writer bounds for the limiting λ value equal to 3.9444 and $\alpha = 0.95$.

Since CVaR is a more conservative measure of risk compared to the expected value of negative terminal positions, the equilibrium is reached at a smaller value of λ for a high confidence level α, e.g. 0.95.

8 Conclusions

We studied the problem of pricing and hedging contingent claims in incomplete markets in a multi-period linear optimization (discrete-time, finite probability space) framework. We developed an extension of the concept of no-arbitrage pricing (sufficiently attractive expected gains) based on expected positive and negative final wealth positions, which allow to obtain arbitrage only in the limit as a risk aversion parameter tends to infinity, in the context of discrete time discrete state investment problems. We analyzed the resulting optimization problems using linear programming duality. We showed that the pricing bounds obtained from our analysis are tighter than the no-arbitrage pricing bounds. This result, in line with the Bernardo and Ledoit [1] single period results, was also obtained for a multi-period model in the computationally more tractable linear programming environment. Proportional transaction costs were easily incorporated into our model. We also extended the SAGE concept to include the Conditional Value-at-Risk measure for measuring losses. Our results indicated that for a limiting value of risk aversion parameter that can be computed easily, a unique price for a contingent claim in incomplete markets may be found while different hedging schemes exist for different sides of the same trade.

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