Restricted Robust Optimization for Maximization over Uniform Matroid with Interval Data Uncertainty

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Abstract

For the problem of selecting $p$ items with uncertain (interval) objective function coefficients so as to maximize total profit (maximization over uniform matroid) we introduce the $r$-restricted robust deviation criterion or the $r$-robust solution for short. We seek solutions that minimize the $r$-restricted robust deviation. This new criterion increases the modeling power of the robust deviation (minmax regret) criterion by reducing the level of conservatism of the robust solution. It is shown that $r$-restricted robust deviation solutions can be computed efficiently. Results of extensive computational experiments and comparisons with absolute robustness, robust deviation and restricted absolute robustness criteria are reported and discussed.

Key words. Maximization over uniform matroid, robust optimization with interval objective function coefficients, $r$-restricted robust deviation.

1 Introduction

The purpose of this paper is to introduce a new measure of robustness called the $r$-restricted robust deviation, and investigate its applicability within the context of a well-known problem from combinatorial optimization, namely the problem of selecting $p$ items out of $n$ so as to maximize total profit. It is assumed that the profit coefficients in the objective function of the problem are uncertain in the sense that they each can assume any value within a finite interval. This type of problems has been introduced and investigated in a series of papers and a monograph by Kouvelis and Yu (and co-authors) [13, 17, 25, 26]. Subsequent contributions include [2, 3, 4, 5, 12, 16, 18, 19, 21, 20, 22, 24] although the list is by no means exhaustive. The unifying theme in all these references is the fact that they all treat one or several well-known combinatorial optimization problems e.g., minimum spanning tree, shortest path etc. where the data are not known with precision. It is usually assumed that the data behave according to some finite or infinite scenarios, or that the data elements can assume any value in some interval. As new concepts of optimality were needed for such situations, the contributors to this area proposed to seek a solution that minimizes (resp. maximizes) some measure of worst performance, i.e., a solution that makes the maximum (resp. the minimum) of a performance measure minimum (resp. maximum). This paradigm gave rise to the concepts of absolute robustness (also known as minmax criterion), and robust deviation (or minmax regret criterion) and robust deviation minimization. There are different definitions of robust optimization problems in the literature [6, 7, 8, 9, 10, 11, 14, 15] although one way or another these approaches also boil down to a minmax or maxmin optimization context. In particular, the approach advocated by Ben-Tal and Nemirovski [7, 8, 9] extensively studied convex

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optimization problems under ellipsoidal data uncertainty although it is not immediately applicable to discrete optimization. Another model by Bertsimas and Sim [10, 11] adopts the interval model of uncertainty and proposes a restricted version of absolute robustness criterion although this connection is overlooked by the authors. Their point of departure is to limit the conservatism of the robust solution by arguing that it is quite unlikely that all data elements will assume their worst possible values simultaneously whereas both the absolute robustness and robust deviation criteria seek solutions for such a contingency.

The picture emerging from these references can be summarized as follows. In the scenario model of uncertainty, and already with the absolute robustness criterion, even the simplest combinatorial optimization problems become intractable. This observation motivated Bertsimas and Sim to develop their restricted absolute robustness paradigm. Another distinguishing feature of the Bertsimas-Sim contribution is the ability to control the conservatism of the robust solutions by restricting its protective power only to a subset of all possible contingencies and establishing a probabilistic guarantee on the remaining events for which no protection is offered.

Using the robust deviation criterion under the scenario model of uncertainty which is a conservative criterion also leads to intractable problems in general. Under the interval model of uncertainty, the picture is somewhat blurred with a mix of positive and negative results. One positive result about tractability in this direction was obtained by Averbakh [2], and further improved by Conde [12], which constitute the starting point of the present paper. Inspired by the work of Bertsimas and Sim, we develop a restricted version of the robust deviation criterion using the problem of Averbakh and Conde, namely, the problem of selecting $p$ elements out of $n$ elements so as to maximize total profit. This problem is also known as the problem of maximization over a uniform matroid and is solvable by a simple procedure in $O(n)$ time (see [12]) if data are known with certainty. Under the interval model of uncertainty of the objective function coefficients, and using the robust deviation criterion, Averbakh gave a polynomial time algorithm, which was improved recently by Conde. In the present paper, we derive the $r$-restricted robust deviation version of the problem and show that it is solvable in polynomial time as well. The main inspiration of this work is the contribution of Bertsimas and Sim whereby we develop a restricted version of the robust deviation criterion in the aim of limiting its conservatism. Although we do not have probabilistic guarantees unlike previous papers in the literature, we compute empirical probabilities, and compare the performances of the absolute robust, restricted absolute robust (Bertsimas-Sim) and the robust deviation solutions with the $r$-restricted robust deviation solution on a large number of randomly generated problem instances.

The rest of the paper is organized as follows. In Section 2 we give a background on robustness criteria illustrated on the maximization problem over uniform matroid. In Section 3 we formulate the $r$-restricted robust problem. In Section 4 we establish its polynomial solvability. Section 5 is devoted to a description of our numerical tests, results and their discussion. Finally, we conclude in Section 6.

## 2 Background

Let a discrete ground set $N$ of $n$ items be given. Denote by $F$ the set of feasible solutions. Consider the following problem

\[
(P) \quad \max \sum_{i \in N} c_i x_i \\
\text{s.t. } x \in F
\]  

(1)
It is assumed that the objective function coefficient of \( i \in N \) denoted by \( c_i \) is not known with certainty, but it is known that it takes a value in the interval \([l_i, u_i]\). For \( i \in N \), we define \( w_i = u_i - l_i \) and assume that \( w_i \geq 0 \).

Let \( S \) denote the set of scenarios. The set \( S \) is the Cartesian product of all intervals. For \( s \in S \), \( c(s) \) denotes the vector of objective function coefficients in scenario \( s \).

Following Kouvelis and Yu [17], we have the following definitions.

**Definition 1** The worst performance for \( x \in F \) is \( a_x = \min_{s \in S} c(s)^T x \). A solution \( x^* \) is called an absolute robust solution if \( x^* \in \arg\max_{x \in F} a_x \). The problem of finding an absolute robust solution is called the absolute robust problem (AR).

The problem AR can be solved easily by solving problem \( P \) for the scenario \( s \) such that \( c(s)_i = l_i \) for all \( i \in N \).

**Definition 2** The robust deviation for \( x \in F \) is \( d_x = \max_{s \in S} (\max_{y \in F} c(s)^T y - c(s)^T x) \). A solution \( x^* \) is called a robust deviation solution if \( x^* \in \arg\min_{x \in F} d_x \). The problem of finding a robust deviation solution is called the robust deviation problem (RD).

Problem RD has received more attention. It has indeed led to problems which are interesting from the modeling and computational point of view (see e.g. [1, 5, 3, 4, 2, 12, 21, 20, 22, 24, 27]).

Both robustness concepts are based on a worst case analysis. They assume that the worst scenario is likely to happen. However, in most practical situations, the probability that all parameters take their worst possible values at the same time may be very small. One may be interested in solutions that are robust when at most a fixed number of parameters take their worst possible values.

Now, following Bertsimas and Sim [10, 11] we give our definition of restricted robust problems.

For \( 1 \leq r \leq n \), define \( S(r) = \{ s \in S : c(s)_i < u_i \) for at most \( r \) items\}.

**Definition 3** The \( r \)-restricted worst performance for \( x \in F \) is \( a^r_x = \min_{s \in S(r)} c(s)^T x \). A solution \( x^* \) is called an \( r \)-restricted absolute robust solution if \( x^* \in \arg\max_{x \in F} a^r_x \). The problem of finding an \( r \)-restricted absolute robust solution is called the \( r \)-restricted absolute robust problem (\( r \)-RAR).

In cases where the feasible set \( F \) of the generic problem \( P \) is a set described by affine inequalities or when \( F \) is a discrete set, Bertsimas and Sim [10, 11] show that whenever \( P \) can be solved in polynomial time, \( r \)-RAR can also be solved in polynomial time.

Now, we can define the problem of interest for the present paper.

**Definition 4** The \( r \)-restricted robust deviation for \( x \in F \) is

\[
d^r_x = \max_{s \in S(r)} (\max_{y \in F} c(s)^T y - c(s)^T x).
\]

A solution \( x^* \) is called an \( r \)-restricted robust deviation solution if \( x^* \in \arg\min_{x \in F} d^r_x \). The problem of finding an \( r \)-restricted robust deviation solution is called the \( r \)-restricted robust deviation problem (\( r \)-RRD).

It is easy to see the following relationship between these problems:

**Proposition 1** For \( x \in F \), \( a^r_x = a_x \) and \( d^r_x = d_x \).
This proposition has the following consequences: problem $AR$ is a special case of problem $r$-$RAR$. Once we know that $r$-$RAR$ is polynomially solvable, it is no surprise that $AR$ is polynomially solvable.

Similarly, problem $RD$ is a special case of problem $r$-$RRD$. Hence, if we know that $RD$ is NP-hard, we can immediately conclude that $r$-$RRD$ is NP-hard. Unfortunately, the robust deviation counterpart of even some easy problems are known to be NP-hard. Examples are shortest path problem (see [27]) and minimum spanning tree problem (see [1]). So the $r$-$RRD$ counterparts of these problems are also NP-hard.

Averbakh [2] proved that problem $RD$ in the setting of maximization over a uniform matroid is a polynomially solvable problem. One of the main questions in this paper is whether $r$-$RRD$ in the same setting is also polynomially solvable. Before investigating the answer to this question, we derive the formulations of problems $RD$ and $r$-$RAR$ for maximization over a uniform matroid. Let $p \leq n$. The deterministic problem is defined as follows:

\[(P) \quad \text{max} \sum_{i \in N} c_i x_i \]
\[\text{s.t.} \quad \sum_{i \in N} x_i = p \quad (2)\]
\[x_i \in \{0, 1\} \quad \forall i \in N. \quad (3)\]

Let $F$ denote the set of feasible solutions. Problem $AR$ can be solved by taking $c_i = l_i$ for all $i \in N$ in the above formulation. To be able to formulate problem $RD$, we need the following result which was shown in [2, 12]:

$$d_x = \max_{y \in F} \left( \sum_{i \in N} u_i - w_i x_i \right) y_i - \sum_{i \in N} l_i x_i.$$

As $\text{conv}(F) = \{ y \in \mathbb{R}_+^n : \sum_{i \in N} y_i = p, y_i \leq 1 \quad \forall i \in N \}$, by the Strong Duality Theorem (SDT) of Linear Programming (LP), we have

$$d_x = \min_{(\lambda, \gamma) \in \Lambda} \left( p \lambda + \sum_{i \in N} \gamma_i \right) - \sum_{i \in N} l_i x_i$$

where $\Lambda = \{ (\lambda, \gamma) \in \mathbb{R}^{n+1} : \lambda + \gamma_i \geq u_i - w_i x_i \text{ and } \gamma_i \geq 0 \quad \forall i \in N \}$. Therefore $RD$ is formulated as:

\[(RD) \quad \text{min} \quad p \lambda + \sum_{i \in N} \gamma_i - \sum_{i \in N} l_i x_i \]
\[\text{s.t.} \quad (2) \text{ and } (3)\]
\[\lambda + \gamma_i \geq u_i - w_i x_i \quad \forall i \in N\]
\[\gamma_i \geq 0 \quad \forall i \in N.\]

The above problem was shown to be polynomially solvable in [2, 12].

Let $1 \leq r \leq n$ and $Z(r) = \{ z \in \{0, 1\}^n : \sum_{i \in N} z_i \leq r \}$. For $x \in F$, the $r$-restricted worst performance can be computed as

$$a^r_x = \sum_{i \in N} u_i x_i - \max_{z \in Z(r)} \sum_{i \in N} w_i z_i x_i.$$
As \( \text{conv}(Z_r) = \{ z \in \mathbb{R}_+^n : \sum_{i \in N} z_i \leq r, z_i \leq 1 \ \forall i \in N \} \), by the SDT of LP, we have

\[
a_x^r = \sum_{i \in N} u_i x_i - \min_{(\mu, \gamma) \in M} \mu r + \sum_{i \in N} \gamma_i
\]

where \( M = \{ (\mu, \gamma) \in \mathbb{R}_+^{\gamma + 1} : \mu + \gamma_i \geq w_i x_i \} \). Therefore the \( r\)-RAR is formulated as follows:

\[
(r\text{-RAR}) \quad \max \sum_{i \in N} u_i x_i - \mu r - \sum_{i \in N} \gamma_i
\]

s.t. (2) and (3)

\[
\mu + \gamma_i \geq w_i x_i \quad \forall i \in N
\]

\[
\mu \geq 0
\]

\[
\gamma_i \geq 0 \quad \forall i \in N.
\]

This problem is also polynomially solvable [10].

To conclude this section, we relate the four robust problems in the context of maximization over a uniform matroid. This result is stronger than the general result in Proposition 1.

**Proposition 2** For \( x \in F \), \( a_x^r = a_x^p = a_x \) for all \( r \geq p \) and \( d_x^r = d_x^p = d_x \) for all \( r \geq \min\{p, n-p\} \).

**Proof.** As \( S(1) \subseteq S(2) \subseteq \ldots \subseteq S(n) = S \), for \( x \in F \), we have \( a_1^x \geq a_2^x \geq \ldots \geq a_n^x = a_x \) and \( d_1^x \leq d_2^x \leq \ldots \leq d_n^x = d_x \).

For \( r > p \), let \( a_x^r = \min_{s \in S(\gamma)} c(s)^T x = c(s^*)^T x \). Construct scenario \( s' \) as follows: for \( i \in N \), if \( c(s^*)_i < u_i \) and \( x_i = 0 \), then \( c(s')_i = u_i \) and \( c(s')_i = c(s^*)_i \) otherwise. Then \( s' \in S(p) \) and \( c(s^*)^T x = c(s')^T x \). As \( a_x^p = \min_{s \in S(p)} c(s)^T x \leq c(s')^T x \), we have \( a_x^p \leq a_x^r \). We also know that \( a_x^p \geq a_x^r \) since \( r > p \). So \( a_x^p = a_x^r \) for all \( r \geq p \).

Let \( \overline{p} = \min\{p, n-p\} \). For \( r > \overline{p} \), let \( d_x^r = \max_{s \in S(\gamma)} c(s)^T x - c(s^*)^T x \). Construct scenario \( s' \) as follows: for \( i \in N \), if \( c(s^*)_i < u_i \) and \( x_i = 0 \) or \( y_i = 1 \), then \( c(s')_i = u_i \) and \( c(s')_i = c(s^*)_i \) otherwise. Then \( s' \in S(\overline{p}) \) and \( c(s^*)^T y^* - c(s^*)^T x \leq c(s')^T y^* - c(s')^T x \). So \( d_x^r \leq c(s^*)^T y^* - c(s')^T x \). As \( d_x^p \geq c(s^*)^T y^* - c(s')^T x \), we have \( d_x^p \leq d_x^r \).

Together with \( d_x^r \geq d_x^p \), this implies that \( d_x^r = d_x^p \) for \( r > \overline{p} \). \( \square \)

### 3 Structural Results and Formulation

In this section we present a MIP formulation of the \( r\)-restricted robust problem after some intermediate results.

**Proposition 3** For \( x \in F \),

\[
d_x^r = \max_{\{z \in Z(r) : y_i + z_i \leq 1 \ \forall i \in N\}} \left( \max_{y \in F} \sum_{i \in N} u_i y_i - \sum_{i \in N} (u_i - w_i z_i) x_i \right). 
\]

**Proof.** Let \( s^* \) and \( y^* \) be such that \( d_x^r = c(s^*)^T y^* - c(s^*)^T x \) and define \( z^* \) and \( v^* \) as follows: for \( i \in N \), if \( c(s^*)_i < u_i \) then \( z_i^* = 1 \) and \( v_i^* = u_i - c(s^*)_i \) and if \( c(s^*)_i = u_i \) then \( z_i^* = 0 \) and \( v_i^* = 0 \).

Consider the scenario \( s' \) such that \( c(s')_i = l_i \) if \( z_i^* = 1 \) and \( y_i^* = 0 \) and \( c(s')_i = u_i \) otherwise.
Theorem 1

Problem $r$-RRD can be formulated as follows:

\[
(f_{x}^{r}) = \min p\lambda + r\mu + \sum_{i \in N} \gamma_{i} - \sum_{i \in N} u_{i}x_{i}
\]

s.t. (2) and (3)

\[
\lambda + \gamma_{i} \geq u_{i} \quad \forall i \in N \tag{9}
\]

\[
\mu + \gamma_{i} \geq w_{i}x_{i} \quad \forall i \in N \tag{10}
\]

\[
\mu \geq 0 \quad \forall i \in N \tag{11}
\]

\[
\gamma_{i} \geq 0 \quad \forall i \in N. \tag{12}
\]

Proof. Let $F_{x}^{r} = \{(y, z) \in \mathbb{R}_{+}^{2n} : (4) - (8)\}$. We first show that $\text{conv}(F_{x}^{r}) = \{(y, z) \in \mathbb{R}_{+}^{2n} : (4) - (6)\}$. Let $H$ be the matrix of left hand side coefficients of constraints

\[
\sum_{i \in N} y_{i} \leq p
\]

\[
-\sum_{i \in N} y_{i} \leq -p
\]

As $d_{x}^{r} = \max_{s \in S(r)} \left(\max_{y \in F} c(s)^{T} y - c(s)^{T} x\right)$, all above inequalities are satisfied at equality. Hence, there exists an optimal solution where $y_{i} + z_{i} \leq 1$ for all $i \in N$ and the objective function coefficient of item $i$ is at its lower bound $l_{i}$ whenever $z_{i} = 1$. □
and constraints (5) and (6). Let \( e \) be the \( n \)-vector of 1’s, \( 0 \) be the \( n \)-vector of 0’s and \( I \) be the \( n \times n \) identity matrix. Then

\[
H = \begin{pmatrix}
e^T & 0^T \\
-e^T & 0^T \\
0^T & e^T \\
I & I
\end{pmatrix}.
\]

Next we show that \( H \) is totally unimodular (TU). Matrix \( H \) is TU if each collection of columns of \( H \) can be split into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries only 0, +1 and −1 (see Schrijver [23], p.269, Theorem 19.3). Let \( H_1 = \{h_1^1, h_2^1, \ldots, h_n^1\} \) be the set of first \( n \) columns of \( H \) and let \( H_2 = \{h_1^2, h_2^2, \ldots, h_n^2\} \) be the set of last \( n \) columns of \( H \). Given a set \( C \) (we can consider sets instead of collections, since a repeated column is put in two different parts) of columns of \( H \), we can partition \( C \) into two parts \( C_1 \) and \( C_2 \) such that the difference of the sum of the columns in \( C_1 \) and the sum of the columns in \( C_2 \) has components 0, +1 and −1 as follows. Without loss of generality, suppose that \( \{1, 2, \ldots, k\} \) is the set of indices \( i \) such that \( h_i^1 \in C \) and \( h_i^2 \notin C \); \( \{k + 1, k + 2, \ldots, l\} \) is the set of indices \( i \) such that \( h_i^1 \notin C \) and \( h_i^2 \in C \). For \( j = 1, 2, \ldots, k \), if \( j \) is odd, put \( h_j^1 \) to \( C_1 \) and \( h_j^2 \) to \( C_2 \) and if \( j \) is even, put \( h_j^1 \) to \( C_2 \) and \( h_j^2 \) to \( C_1 \). For \( j = 1, 2, \ldots, l - k \), if \( j \) is odd, put \( h_j^1 \) to \( C_2 \) and if \( j \) is even, put \( h_j^1 \) to \( C_1 \). For \( j = 1, 2, \ldots, m - l \), if \( j \) is odd, put \( h_j^2 \) to \( C_1 \) and if \( j \) is even, put \( h_j^2 \) to \( C_2 \).

As \( H \) is TU and the right hand side vector is integral, \( \{(y, z) \in \mathbb{R}^{2n}_+ : (4) - (6)\} \) is an integral polytope (see Schrijver [23], p.268, Corollary 19.2a).

As a consequence of this observation, for a given \( x \), the \( r \)-restricted robust deviation can be computed by solving a linear program. Associate dual variables \( \lambda \) to constraint (4), \( \mu \) to constraint (5) and \( \gamma_i \) to constraint (6) for all \( i \in N \). Then SDT of LP implies that

\[
f^*_x = \min p\lambda + r\mu + \sum_{i \in N} \gamma_i \quad \text{s.t.} \quad (9) - (12).
\]

This concludes the proof. \( \Box \)

It is easy to verify that when \( r = p \) the formulation of Theorem 1 is transformed to RD of [12].

## 4 Solvability Status of the \( r \)-Restricted Robust Problem

In this section we establish a key property of the optimal solution set of the problem \( r \text{-RRD} \). More precisely, we show below that there exists an optimal solution where the search space for \( \lambda \) and \( \mu \) is significantly reduced.

Let \( L = \{l_1, l_2, \ldots, l_n\} \), \( U = \{u_1, u_2, \ldots, u_n\} \) and \( W = \{w_1, w_2, \ldots, w_n\} \).

**Theorem 2** There exists \((x^*, \lambda^*, \mu^*, \gamma^*)\) optimal for \( r \text{-RRD} \) such that the following statements are true:

a. Either \( \lambda^* - \mu^* \in L \) or \( \mu^* \in W \) and \( \lambda^* \in U \).

b. If \( \mu^* > 0 \) then either \( \lambda^* \in U \) or \( \mu^* \in W \).
Proof. Let \((x^*, \lambda^*, \mu^*, \gamma^*)\) be an extreme point optimal solution. Without loss of generality, assume \(x_i^* = 1\) for \(i \in \{1, \ldots, p\}\) and \(x_i^* = 0\) otherwise. Then,

\[
\gamma_i^* = \max\{w_i - \mu^*, u_i - \lambda^*, 0\} \quad \text{for} \quad i \in \{1, \ldots, p\},
\]

and

\[
\gamma_i^* = \max\{u_i - \lambda^*, 0\} \quad \text{for} \quad i \in \{p + 1, \ldots, n\}.
\]

Define \(A\) such that \(A = \{i \in \{1, \ldots, p\} : w_i - \mu^* > u_i - \lambda^*\}\). We subdivide \(A\) into \(A_1, A_2\) and \(A_3\) such that \(A_1 = \{i \in A : w_i - \mu^* > 0\}, A_2 = \{i \in A : w_i - \mu^* = 0\}\) and \(A_3 = \{i \in A : w_i - \mu^* < 0\}\). Let \(B = \{i \in \{1, \ldots, p\} : w_i - \mu^* = u_i - \lambda^*\}\) subdivided into \(B_1 = \{i \in B : w_i - \mu^* = u_i - \lambda^* > 0\}, B_2 = \{i \in B : w_i - \mu^* = u_i - \lambda^* = 0\},\) and \(B_3 = \{i \in B : w_i - \mu^* = u_i - \lambda^* < 0\}\). We also have \(C = \{i \in \{1, \ldots, p\} : w_i - \mu^* < u_i - \lambda^*\}\) partitioned into \(C_1 = \{i \in C : u_i - \lambda^* > 0\}, C_2 = \{i \in C : u_i - \lambda^* = 0\},\) and \(C_3 = \{i \in C : u_i - \lambda^* < 0\}\). We make the same definition for the indices in \(\{p + 1, \ldots, n\}\). Namely, let \(D = \{i \in \{p + 1, \ldots, n\} : u_i - \lambda^* > 0\}, E = \{i \in \{p + 1, \ldots, n\} : u_i - \lambda^* = 0\}\) and \(F = \{i \in \{p + 1, \ldots, n\} : u_i - \lambda^* < 0\}\).

First for the proof of part a, assume neither \(\lambda^* - \mu^* \in L\) nor \(\mu^* \in W\). This implies \(A_2 = B = \emptyset\). Now, we have \(\gamma_i^* = w_i - \mu^*\) for \(i \in A_1, \gamma_i^* = 0\) for \(i \in A_3, \gamma_i^* = u_i - \lambda^*\) for \(i \in C_1, \gamma_i^* = u_i - \lambda^* = 0\) for \(i \in C_2, \gamma_i^* = u_i - \lambda^* = 0\) for \(i \in C_3, \gamma_i^* = u_i - \lambda^* = 0\) for \(i \in D, \gamma_i^* = u_i - \lambda^* = 0\) for \(i \in E\) and finally \(\gamma_i^* = 0\) for \(i \in F\). Let \((x^*, \lambda^*, \mu^*, \gamma_A^*, \gamma_A^*, \gamma_C^*, \gamma_C^*, \gamma_D^*, \gamma_D^*, \gamma_E^*, \gamma_E^*)\) be the vectorial description of the current solution. For a given set \(S\) and some \(\epsilon\), let \(\epsilon_S\) be a vector of dimension of \(|S|\) with all entries equal to \(\epsilon\). Then, both

\[
(x^*, \lambda^*, \mu^* + \epsilon, \gamma_A^* + \epsilon, \gamma_A^* + \epsilon, \gamma_C^* + \epsilon, \gamma_C^* + \epsilon, \gamma_D^* + \epsilon, \gamma_E^* + \epsilon)
\]

and

\[
(x^*, \lambda^*, \mu^* - \epsilon, \gamma_A^* + \epsilon, \gamma_A^* + \epsilon, \gamma_C^* + \epsilon, \gamma_C^* + \epsilon, \gamma_D^* + \epsilon, \gamma_E^* + \epsilon)
\]

are feasible solutions of \(r\)-RRD for small enough \(\epsilon\) which contradicts the extremality of the starting optimal solution.

Now, assume neither \(\lambda^* - \mu^* \in L\) nor \(\lambda^* \in U\). This implies that \(B = C_2 = E = \emptyset\). If we proceed as above, we can conclude easily that both

\[
(x^*, \lambda^* + \epsilon, \mu^*, \gamma_A^*, \gamma_A^* + \epsilon, \gamma_A^* + \epsilon, \gamma_C^* + \epsilon, \gamma_C^* + \epsilon, \gamma_D^* - \epsilon, \gamma_D^* + \epsilon)
\]

and

\[
(x^*, \lambda^* - \epsilon, \mu^*, \gamma_A^*, \gamma_A^* + \epsilon, \gamma_A^* + \epsilon, \gamma_C^* + \epsilon, \gamma_C^* + \epsilon, \gamma_D + \epsilon, \gamma_D + \epsilon)
\]

are again feasible solutions to \(r\)-RRD with small enough \(\epsilon > 0\).

For part b, assume that \(\mu^* > 0\) but neither \(\lambda^* \in U\) nor \(\mu^* \in W\). This implies that \(A_2 = B_2 = C_2 = E = \emptyset\). For \(\epsilon > 0\), both

\[
(x^*, \lambda^* + \epsilon, \mu^* + \epsilon, \gamma_A^* + \epsilon, \gamma_A^* + \epsilon, \gamma_B^* + \epsilon, \gamma_B^* + \epsilon, \gamma_C^* + \epsilon, \gamma_C^* + \epsilon, \gamma_D - \epsilon, \gamma_D + \epsilon)
\]

and

\[
(x^*, \lambda^* - \epsilon, \mu^* - \epsilon, \gamma_A^* + \epsilon, \gamma_A^* + \epsilon, \gamma_B^* + \epsilon, \gamma_B^* + \epsilon, \gamma_C^* + \epsilon, \gamma_C^* + \epsilon, \gamma_D + \epsilon, \gamma_D + \epsilon)
\]

and this is again in contradiction with the extremality of our optimal solution. \(\square\)

An important consequence of this theorem is that the solution of the problem \(r\)-RRD for fixed \(\lambda\) and \(\mu\), boils down to minimization of a piecewise linear function very much like
problem (3) of [12], the evaluation of which can be accomplished in \( O(n) \) time. More precisely for fixed \( \lambda \) and \( \mu \), we solve the following problem

\[
f(\lambda, \mu) = p\lambda + r\mu + \min \sum_{i \in N} \max\{u_i - \lambda, w_i x_i - \mu, 0\} - \sum_{i \in N} u_i x_i \\
\text{s.t. (2) and (3)}. \tag{14}
\]

Therefore, we transformed \( r\text{-RRD} \) into

\[
\min_{\lambda, \mu} f(\lambda, \mu)
\]

with \( f(\lambda, \mu) \) defined above. We solve the latter problem by testing the critical values of \( \lambda \) and \( \mu \) following Theorem 1 as follows. If \( \lambda \in U \) and \( \mu \in W \), then we need to test \( n^2 \) values. Otherwise, \( \lambda - \mu \in L \). If \( \mu = 0 \), then \( \lambda \in L \) and so can take \( n \) different values. Finally, if \( \mu > 0 \), then either \( \lambda \in U \) or \( \mu \in W \), resulting in additional \( 2n^2 \) values to test. We simply pick the \( \lambda \) and \( \mu \) values yielding the smallest objective function value. Therefore, we have proved the following result.

**Theorem 3** Problem \( r\text{-RRD} \) can be solved in \( O(n^3) \) time.

5 Experiments

In this section we provide experimental evidence to the effectiveness of \( r\)-restricted robust deviation criterion on the problem of robust maximization over uniform matroid.

In the paragraphs below, when we speak of an uncertain problem we mean that we fix \( n \) and \( p \) and generate for each objective function coefficient a random interval, i.e., \( l_i \) and \( u_i \) values for each \( i \). In all of the consequent experiments we generate an uncertain problem with \( n = 500 \) randomly where we take the \( l_i \) values uniformly distributed in the interval \((-10000, 10000)\), \( w_i \) values uniformly distributed in the interval \((0, 2000)\) and we obtain \( u_i \) values by simply adding \( w_i \) to \( l_i \) for each \( i \) from 1 to \( n \). We consider a range of values for \( p \). Then by an instance of an uncertain problem we mean a random scenario of objective function coefficients within the prespecified interval while everything else is kept fixed.

We report the outcome of three experiments.

5.1 Experiment 1: Convergence of \( r\text{-RRD} \) to RD

In the first experiment, we aim to demonstrate that the choice of \( r \) is not critical to the performance of the criterion. To measure performance, we compute the robust deviation value, using the optimal solution obtained from solving the problem \( r\text{-RRD} \), i.e., the \( r \)-robust solution. To conduct this experiment we generate an uncertain problem as explained above. Keeping all data fixed, we vary \( r \) from 1 to \( \min\{p, n - p\} \), and plot the robust deviation \( d_x \) values as in Definition 2 as a function of \( r \). This test was repeated several times with different uncertain problems and different values of \( p \). A typical result with \( p = 100 \) is shown in Figure 1 where we observe that the robust deviation value obtained from solving \( r\text{-RRD} \) quickly converges to the minmax-regret value already for small values of \( r \). We can reach two conclusions from this experiment. The first one is that the choice of \( r \) is immaterial from a certain value on, where this threshold value is typically reached very quickly. The second conclusion is that once this threshold value of \( r \) is reached, the minmax regret and \( r \)-restricted robust solutions are indistinguishable as far as the smallest value of maximum regret.
is concerned. This is an interesting experimental finding in the sense that by introducing the $r$-restricted deviation we hedged ourselves only to damages limited by the bad behavior of $r$ coefficients. However, full protection with respect to maximum regret is already achieved much earlier, before reaching $r = \min\{p, n-p\}$.

5.2 Experiment 2: Comparison of Relative Performances

Our second experiment consists in comparing the relative performance of the four robustness measures discussed in the present paper. We report results of the second experiment in Figures 2, 3, 4, 5 and Tables 1, 2 and 3 below. The reason for separating the results into four figures and three tables is to avoid encumbering plots unnecessarily with four different solutions, and to see clearly the relative behavior of the $r$-robust solution vis-a-vis different criteria. For Figures 2–5, we first generate an uncertain problem as explained previously with $p$ taking values from the set $\{50, 100, 200, 250\}$. Then we randomly generate what we call “extreme” problem instances in order to be able to observe a distinct behavior of the $r$-robust and the robust deviation solutions, which is usually impossible to obtain without such extreme instances. These instances are generated as follows. For a fixed $r$, we take objective function coefficients at their lower bounds with probability equal to $\frac{r}{n}$ and at their upper bounds with probability equal to $1 - \frac{r}{n}$. We repeat this 50 times, thus obtaining 50 problem instances for fixed $p$ and $r$. Then we compute the $r$-restricted robust solution and the robust deviation solution. We calculate the deviations of these solutions from the optimal values of each of the 50 instances. We take the average of these 50 observations for both the $r$-robust solution and the robust deviation solution, respectively. Therefore we obtain two points for our plots for a fixed value of $r$. Repeating this for values of $r$ ranging from 1 to $p$, we obtain an entire plot. We observe from our four plots that for small values of $r$, the $r$-robust solution is somewhat more robust in such extreme scenario compared to the robust deviation solution whereas after a certain $r$ value the two solutions behave identically.

As the above results show that the $r$-restricted robust solution behaves like the minmax regret solution already for small values of $r$, in the remainder of the second experiment, we compare the performances of the $r$-robust deviation solution (by solving $r$-$RRD$) for a fixed value of $r$, the $r$-restricted robust absolute solution using the same value of $r$, and the
Figure 2: Average percentage deviations of robust deviation and $r$-robust solutions from the optimal value for extreme problem instances: $p = 50$

Figure 3: Average percentage deviations of robust deviation and $r$-robust solutions from the optimal value for extreme problem instances: $p = 100$
Figure 4: Average percentage deviations of robust deviation and \( r \)-robust solutions from the optimal value for extreme problem instances: \( p = 200 \)

Figure 5: Average percentage deviations of robust deviation and \( r \)-robust solutions from the optimal value for extreme problem instances: \( p = 250 \)
absolute robust solution of Definition 1 for 50 uncertain problems in Tables 1, 2 and 3. For each uncertain problem, fixing $p$ and $r$ we first compute these three solutions. With these three solutions at hand, we generate 500 random instances. We find the optimal value for each of the 500 problem instances, and compute the percentage error in objective function value corresponding to each of the three solutions with respect to the random instance’s optimal value. We compute the $\ell_1$ and $\ell_\infty$ norms of this error vector of dimension 500.

We use three probability distributions to generate objective function coefficients randomly: (1) the objective function coefficients are distributed according to a Normal law where the interval $[l_i, u_i]$ for coefficient $i$ corresponds to a 95%-quantile (see Table 1), (2) the objective function coefficients are distributed according to a uniform law in the interval $[l_i, u_i]$ (see Table 2) and (3) the objective function coefficient $i$ takes value $l_i$ and $u_i$ with equal probability of one-half; this is referred to as the two-point distribution (see Table 3).

We repeat this procedure for 50 randomly generated uncertain problems. Finally, each entry of these tables corresponds to the mean value of these $\ell_1$ and $\ell_\infty$ norms of percentage errors over 50 uncertain problems.

The results clearly show a superiority of the $r$-restricted robust deviation solution to absolute robust and restricted absolute robust solutions.

### 5.3 Experiment 3: Comparison of Absolute Performances

Since the purpose of computing an $r$-restricted robust deviation solution is to protect ourselves against random fluctuations in the objective function coefficients while we try to limit the over-conservatism of the solution, it is meaningful to ask the following question: what is the probability (under an assumed distribution of the objective function coefficients) that the robust solution fails to give a satisfactory performance? The smaller this probability, the higher the protection offered by the $r$-restricted robust deviation solution. This question is a legitimate question since we are only offering full protection for a subset of all events defined by the set $S(r)$. This question is addressed analytically in the references [10, 11] where the authors derive upper bounds on the probability of unsatisfactory performance for
Table 2: $\ell_1$ and $\ell_\infty$ norms of error vectors for three different robustness criteria averaged over 50 uncertain problems tested against 500 uniformly distributed randomly generated instances.

<table>
<thead>
<tr>
<th>p,r</th>
<th>$r$-RRD</th>
<th>$r$-RAR</th>
<th>AR</th>
<th>$r$-RRD</th>
<th>$r$-RAR</th>
<th>AR</th>
</tr>
</thead>
<tbody>
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<td>2,327</td>
<td>2,442</td>
<td>0,007</td>
<td>0,009</td>
<td>0,010</td>
</tr>
<tr>
<td>50,25</td>
<td>1,476</td>
<td>1,819</td>
<td>2,442</td>
<td>0,007</td>
<td>0,008</td>
<td>0,010</td>
</tr>
<tr>
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<td>2,114</td>
<td>2,442</td>
<td>0,007</td>
<td>0,009</td>
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</tr>
<tr>
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<td>1,273</td>
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<td>0,005</td>
<td>0,006</td>
</tr>
<tr>
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<tr>
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<td>1,294</td>
<td>1,534</td>
<td>0,004</td>
<td>0,006</td>
<td>0,006</td>
</tr>
<tr>
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<td>0,952</td>
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<td>0,004</td>
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</table>

Table 3: $\ell_1$ and $\ell_\infty$ norms of error vectors for three different robustness criteria averaged over 50 uncertain problems tested against 500 two-point distributed randomly generated instances.

<table>
<thead>
<tr>
<th>p,r</th>
<th>$r$-RRD</th>
<th>$r$-RAR</th>
<th>AR</th>
<th>$r$-RRD</th>
<th>$r$-RAR</th>
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<td>0,011</td>
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<td>3,251</td>
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<td>0,006</td>
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</table>
the $r$-restricted absolute robust problem under some general assumptions on the distribution of random parameters. An unsatisfactory performance in this context is defined as the occurrence of an objective function value at the $r$-restricted robust deviation solution smaller than the $r$-restricted worst performance (as in Definition 3) of the same solution. The reason for using Definition 3 in place of Definition 4 is that we are interested in an absolute performance in this experiment. In the light of the above remarks, our third and final experiment consists in computing an empirical probability of unsatisfactory performance for the $r$-restricted robust deviation solution as a function of $r$ for a problem with fixed $n$ and $p$. For comparison, this empirical probability is also computed for the $r$-restricted absolute robust solution of Definition 3.

We conduct the experiment as follows. We randomly generate uncertain problems as explained with $p = 50, 100, 200$ and 250 and solve the corresponding $r$-restricted absolute robust problem and the $r$-restricted robust deviation problem. Then, using each one of the three distributions as described in the previous subsection for Experiment 2, we generate 10000 random objective function vectors and count the occurrences of unsatisfactory trials for all robust solutions. This is repeated for increasing values of $r$ from 1 to $p$. This experiment is also useful in guiding the choice of $r$ for a given problem. While we clearly do not want to choose $r$ too large, it is certainly beneficial from a robustness point of view to choose $r$ in the range where the probability of unsatisfactory performance is very small or even nil. The results of this experiment are reported in Figures 6–8. To avoid too many plots, we give 4 continuous curves and 4 dashed curves in each plot corresponding respectively to the $r$-restricted robust solution and the $r$-restricted absolute robust solution for increasing values of $p$ from 50 to 250. We notice that the the critical value of $r$ (where the probability of unsatisfactory performance vanishes) for a fixed problem does not vary so much according to the probability distribution as it does with the choice $p$. For larger $p$, e.g., $p = 200, 250$, the critical value is situated at a value $\alpha p$ where $\alpha$ is slightly above $1/3$, e.g. 0.36 or 0.37. This critical value becomes somewhat larger as $p$ gets smaller, e.g., for $p = 100$ it is around 41, while for $p = 50$, it seems to be 25. A noticeable increase occurs when we use the two-point distribution for the objective function coefficients.

We note that the performances of the $r$-restricted absolute robust solution and that of the $r$-restricted robust deviation solutions are quite close, with a slight superiority of the $r$-restricted robust deviation solution in all cases.

6 Conclusions

In this paper we considered a non-trivial generalization of the robust deviation criterion by introducing the $r$-restricted robust deviation criterion for maximization problem over uniform matroid with uncertain objective function coefficients. We have shown how to transform the problem into a linear mixed-integer programming problem, and proved that this problem can be solved in $O(n^3)$ time. We reported results of extensive computational experiments with thousands of uncertain test problems where we compared the performance of the $r$-restricted robust deviation solution to earlier proposals such as the robust deviation solution, and the $r$-restricted absolute robust solution. The results revealed that the $r$-restriction of the robust deviation solution, in spite of the fact that it is less conservative, does not lead to a decrease in robustness and shows a superior behavior to previously proposed robustness concepts.
Figure 6: Empirical probability of unsatisfactory performance using normally distributed objective function coefficients as a function of $r$.

Figure 7: Empirical probability of unsatisfactory performance using uniformly distributed objective function coefficients as a function of $r$. 
Figure 8: Empirical probability of unsatisfactory performance using two point distributed objective function coefficients as a function of $r$.

References


