On Robust Quadratic Hedging of Contingent Claims in Incomplete Markets under Ambiguous Uncertainty

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Abstract

We consider the quadratic hedging problem for a contingent claim in a single period framework where the price of the underlying asset follows rather simple stochastic models. However, it is assumed that there is ambiguity in the specification of the models, i.e., that the parameters of the models are subject to uncertainty. We treat these uncertainties in the framework of robust optimization and investigate tractability of the resulting robust quadratic hedging problems. Numerical simulations show clearly that taking into account ambiguity in the specification of stochastic parameters results in more robust hedging policies.

Key words. Quadratic hedging, robust optimization, scenario approximation, contingent claims, uncertainty, ambiguity.

AMS subject classifications. 91B28, 90C90.

1 Introduction

The purpose of the present paper is to investigate the single period quadratic hedging problem of contingent claims in incomplete markets under some simple presumed models of asset price that are plagued with the problem of ambiguity. The term "ambiguity" in this context is intended to mean that the stochastic models governing the movement of asset prices have parameters that are not known with certainty. Under this ambiguity of stochastic asset price models, we investigate min-max type robust, optimal quadratic hedging policies based on local risk minimization and associated computational issues.

We consider the quadratic hedging problem in a single period setup in incomplete markets [7]. Our economy consists of a risky asset X with price X_t at times $t = 0, 1$ and a riskless asset (called bond or bank account). The price X_1 of the stock at time $t = 1$ is a random variable measurable

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with respect to the information accumulated up to time $t = 1$, while X_0 is known at time $t = 0$. We can assume w.l.o.g. that the bond price is equal to one. We denote by ξ the shares of X bought at time $t = 0$, and by η_0 the amount invested in the riskless asset. The value of the portfolio at time t is given by

$$V_t = \xi X_t + \eta_t. \tag{1}$$

The problem is to determine an optimal hedge for a contingent claim with random pay-off H at time $t = 1$, i.e. choose values of ξ and η_0, η_1 to minimize the cost of the hedging strategy.

Since one can always set $V_1 = H$, this gives $\eta_1 = H - \xi X_1$ which implies that the hedging strategy is non-self-financing. Let us denote the cumulative cost of the strategy by C_t. Then we have

$$C_0 = V_0 = \xi X_0 + \eta_0.$$

The additional cost involved in passing from time period 0 to time period 1 is given by

$$C_1 - C_0 = \eta_1 - \eta_0 = H - V_0 - \xi \Delta X$$

where $\Delta X = X_1 - X_0$. The problem is now to determine the values V_0 and ξ so as to minimize the expected quadratic additional cost of the trading strategy

$$\min_{V_0,\xi} R := \mathbb{E}[(H - V_0 - \xi \Delta X)^2].$$

The solution to this problem is contained in [10], and is as follows:

$$\xi = \frac{Cov(H, X_1)}{Var[X_1]} \tag{2}$$

$$V_0 = \mathbb{E}[H] - \xi \mathbb{E}[\Delta X]. \tag{3}$$

It also true that at an optimal solution we have

$$C_0 = \mathbb{E}[C_1]$$

which implies that the optimal portfolio strategy's cumulative costs form a *mean*-self-financing strategy. For a comprehensive survey of quadratic hedging, the reader is directed to [11]. Three papers related with the present study are due to Avellaneda *et al.* and to Ahn *et al.* [1, 2, 3] where the authors develop robust hedging strategies for option pricing if the volatility of the asset price is misspecified, but stays in an interval of uncertainty. The first paper shows that the derivative asset price in a Black-Scholes framework under an interval uncertainty model for the volatility is described as the solution of a nonlinear PDE. The second and third papers use exponential utility function pricing to obtain robust hedging strategies for Black-Scholes hedging strategies in [2], and extend their results to more general hedging strategies in [3]. Our contribution differs from these two papers in that we address simultaneous uncertainty in both the drift and volatility terms (while we can certainly address uncertainty in volatility only), and use expected quadratic cost criterion as hedging mechanism.

In the present paper we will be concerned with the application of robust optimization techniques to the quadratic hedging problem under ambiguous stochastic models of asset prices. We treat some simple cases of ambiguity assuming independent ambiguity under two different stochastic asset price models in Section 2. Section 3 assumes a dependence among ambiguous parameters and investigates the structure of resulting optimization problems. The models obtained in Section 2 are in fact special cases of the more general model of Section 3. Section 4 is devoted to the numerical solution of the models, and Section 5 to the results of computational experiments.

2 A Simple Robust Model for European Calls: Interval Uncertainty

In this section we will introduce the robust hedging problem using two models of uncertainty for the stock price. The models presented below are in fact special cases of a more general model to be given in Section 3.

2.1 A Simple Stochastic Model

We will assume throughout this section that the movement of the asset price is governed by the model

$$X_1 = X_0(1 + \mu + \sigma\omega) \tag{4}$$

where μ and σ are parameters, and ω is a random variable with density function $\mathcal{P}(\omega)$ and support $\Omega \subseteq \mathbb{R}$. Therefore, we have $\Delta X(\omega) = X_0(\mu + \sigma\omega)$. For European call options with strike K, we have

$$H(\omega) = (X_1(\omega) - K)_+ = [X_0(1 + \mu + \sigma\omega) - K]_+. \tag{5}$$

We can evaluate the function $\mathbb{E}[(H - V_0 - \xi\Delta X)^2]$ under the assumption that the random variable ω follows the standard Normal distribution. This expression is useful in numerical computations.

Lemma 1 *If ω follows a standard Normal distribution, X_1 is given by (4) and H by (5), the function $\mathbb{E}[(H - V_0 - \xi\Delta X)^2]$ is given by*

$$A + B(-2V_0 - 2\xi\mu X_0) + C(-2\xi X_0\sigma) + V_0^2 + 2V_0\xi X_0\mu + \xi^2(X_0^2\mu^2 + X_0^2\sigma^2)$$

with

$$A = [X_0^2(1+2\mu+\mu^2) - 2K(1+\mu) + K^2](1 - F_\omega(b)) + [X_0^2(2\sigma+2\mu\sigma) - 2KX_0\sigma]f_\omega(b) + X_0^2\sigma^2(1 - F_\omega(b) + bf_\omega(b))$$

$$B = [X_0(1 + \mu) - K](1 - F_\omega(b)) + X_0\sigma f_\omega(b)$$

$$C = [X_0(1 + \mu) - K]f_\omega(b) + X_0\sigma(1 - F_\omega(b) + bf_\omega(b))$$

where f_ω is the standard Normal density, F_ω is the standard Normal cumulative distribution function, and $b = \frac{K}{X_0\sigma} - \frac{1+\mu}{\sigma}$.

Proof: By straightforward calculation, and [12] for the partial moments. ∎

We will assume that the value of μ is not known precisely, but known only to lie between lower bound μ_L and upper bound μ_U. Therefore, we are now facing the robust quadratic hedging problem [RobQHμ]:

$$\min_{V_0,\xi}\max_{\mu_L\le\mu\le\mu_U}\phi(V_0,\xi,\mu):=\mathbb{E}_\omega\left\{[(X_0(1+\mu+\sigma\omega)-K)_+-V_0-\xi X_0(\mu+\sigma\omega)]^2\right\}.$$

Let us now concentrate on the function

$$g_\omega(\mu,\sigma,V_0,\xi):=[(X_0(1+\mu+\sigma\omega)-K)_+-V_0-\xi X_0(\mu+\sigma\omega)]^2$$

as a function of μ (the function g is assumed to be measurable and P-integrable). The term $(X_0(1+\mu+\sigma\omega)-K)_+$ is a convex function of μ since it is the maximum of two affine functions in μ (the function $X_0(1+\mu+\sigma\omega)$ and the identically zero function). Unfortunately, squaring a convex piecewise affine function does not always preserve convexity.

A simple extension of the above is to consider an interval uncertainty case in σ alone. This problem gives us the robust quadratic hedging problem [RobQHσ]:

$$\min_{V_0,\xi}\max_{\sigma_L\le\sigma\le\sigma_U}\phi(V_0,\xi,\sigma):=\mathbb{E}_\omega\left\{[(X_0(1+\mu+\sigma\omega)-K)_+-V_0-\xi X_0(\mu+\sigma\omega)]^2\right\}.$$

Let us again check the function $g_\omega(\mu,\sigma,V_0,\xi)$ this time as a function of σ. The term $(X_0(1+\mu+\sigma\omega)-K)_+$ is a convex function of σ since it is the maximum of two affine functions in σ (the function $X_0(1+\mu+\sigma\omega)$ and the identically zero function). However, for the same reason as in the previous paragraph for μ, the square of a piecewise affine function need not be convex.

2.2 An Exponential Stochastic Asset Price Model

An alternative stochastic model for the stock price would be

$$X_1=X_0\mathrm{e}^{\mu+\sigma\omega}\tag{6}$$

as a first-order approximation of the Brownian motion based stock price movement. The problem RobQHμ would then become (we refer to it as RobQHeμ):

$$\min_{V_0,\xi}\max_{\mu_L\le\mu\le\mu_U}\phi(V_0,\xi,\mu):=\mathbb{E}_\omega\left\{[(X_0\mathrm{e}^{\mu+\sigma\omega)}-K)_+-V_0-\xi X_0(\mathrm{e}^{\mu+\sigma\omega}-1)]^2\right\}.$$

An expression for the objective function is derived using direct calculation and results of [12].

Lemma 2 *If ω follows a standard Normal distribution, X_1 is given by (6) and H by (5), the function $\mathbb{E}[(H-V_0-\xi\Delta X)^2]$ is given by*

$$A+B(-2V_0+2\xi\mu X_0)+C(-2\xi X_0e^\mu)+e^{\sigma^2/2}(2V_0\xi X_0e^\mu-2\xi^2X_0^2e^\mu)+e^{2\sigma^2}(\xi^2X_0^2e^{2\mu})+V_0^2-2V_0\xi X_0+\xi^2X_0^2\mu^2$$

with

$$A=K^2e^{\sigma^2/2}(1-F_\omega(b))+X_0^2e^{\mu+\sigma^2/2}(1-F_\omega(b-2\sigma))-2KX_0e^{\mu+\sigma^2/2}(1-F_\omega(b-\sigma))$$

$$B=X_0e^{\mu+\sigma^2/2}(1-F_\omega(b-\sigma))-K(1-F_\omega(b))$$

$$C=X_0e^{\mu+2\sigma^2}(1-F_\omega(b-2\sigma))-Ke^{\sigma^2/2}(1-F_\omega(b-\sigma))$$

where F_ω is the standard Normal cumulative distribution function, and $b=\frac{1}{\sigma}(\ln(\frac{K}{X_0})-\mu)$.

Unfortunately, the convexity of the function

$$g_\omega(V_0,\xi,\mu,\sigma) = [(X_0 \mathrm{e}^{\mu+\sigma\omega} - K)_+ - V_0 - \xi X_0(\mathrm{e}^{\mu+\sigma\omega} - 1)]^2$$

as a function of μ (or, of σ) is not assured. The first term $(X_0\mathrm{e}^{\mu+\sigma\omega} - K)_+$ is convex in μ since $X_0 > 0$, $\mathrm{e}^{\mu+\sigma\omega}$ is convex in μ and, finally, the maximum of two convex functions (maximum of 0 and $\mathrm{e}^{\mu+\sigma\omega}$) is convex. However, the term $-\xi X_0(\mathrm{e}^{\mu+\sigma\omega} - 1)$ may or may not be convex, destroying the convexity of g as a function of μ. The problem [RobQHeσ] with interval uncertainty in σ is:

$$\min_{V_0,\xi}\ \max_{\sigma_L\le\sigma\le\sigma_U} \phi(V_0,\xi,\mu) := \mathbb{E}_\omega\left\{[(X_0\mathrm{e}^{\mu+\sigma\omega)} - K)_+ - V_0 - \xi X_0(\mathrm{e}^{\mu+\sigma\omega} - 1)]^2\right\},$$

which suffers again from lack of convexity in σ.

A variant of the exponential asset price model would be

$$X_1 = X_0\mathrm{e}^{\mu-\frac{1}{2}\sigma^2+\sigma\omega}. \tag{7}$$

In our experience, as far as experimental results are concerned, there was no discernible difference between the model (6), and (7) above. Therefore, we omit the discussion and results for (7). In the next section we will give a result that characterizes the structure of the solutions for all the models of this section as well as for more general case of joint and dependent uncertainty for μ,σ.

3 A Robust Model with Joint Uncertainty in Parameters

In the present section, the uncertainty is extended to cover the second parameter σ as well as μ in the following set membership description:

$$\begin{pmatrix}\mu\\ \sigma\end{pmatrix} \in \mathcal{U} := \left\{\begin{pmatrix}\mu\\ \sigma\end{pmatrix} \mid \begin{pmatrix}\mu\\ \sigma\end{pmatrix} = \begin{pmatrix}\bar\mu\\ \bar\sigma\end{pmatrix} + Pu : \|u\|_2 \le 1\right\}.$$

We are now facing the robust quadratic hedging problem [RobQH2]:

$$\min_{V_0,\xi}\ \max_{(\mu,\sigma)\in\mathcal{U}} \Phi(V_0,\xi,\mu,\sigma) := \mathbb{E}_\omega\left\{[(X_0(1+\mu+\sigma\omega) - K)_+ - V_0 - \xi X_0(\mu+\sigma\omega)]^2\right\}.$$

Notice that zeroing out the off-diagonal elements of P, we can obtain a separable model of ambiguity in μ,σ. In addition to the off-diagonal element, zeroing out the diagonal element P_{11} results in interval uncertainty in σ alone with the interval $[\bar\sigma - P_{22}, \bar\sigma + P_{22}]$, and conversely for μ. Therefore, all the discussion given in this section also applies to problems of Section 2.

The function

$$g_\omega(\mu,\sigma,V_0,\xi) := [(X_0(1+\mu+\sigma\omega) - K)_+ - V_0 - \xi X_0(\mu+\sigma\omega)]^2$$

assumed to be measurable and $\mathcal{P}$-integrable again fails the convexity test in μ,σ for reasons discussed earlier. Now, we give a result that characterizes the structure of the optimal solutions. Notice that the result is valid for all asset price models (4), (6) and (7), and for all the problems of Section 2.

Proposition 1 *A pair V_0^*, ξ^* is optimal in [RobQH2] if and only if there exist couples $(\mu_1, \sigma_1) \ldots, (\mu_r, \sigma_r)$, $1 \leq r \leq 3$ such that*

$$\max_{(\mu,\sigma)\in\mathcal{U}} \Phi(V_0, \xi, \mu, \sigma) = \Phi(V_0, \xi, \mu_k, \sigma_k),\ k = 1, \ldots, r$$

and r non-negative numbers $\alpha_k \geq 0$, $\sum_{k=1}^r \alpha_k = 1$ such that V_0^*, ξ^* are given by the formulas:

$$\xi^* = \frac{\sum_{k=1}^r \alpha_k Cov_k(H, X_1)}{\sum_{k=1}^r \alpha_k Var_k[X_1]} \tag{8}$$

$$V_0^* = \sum_{k=1}^r \alpha_k \mathbb{E}_k[H] - \xi^* \sum_{k=1}^r \alpha_k \mathbb{E}_k[\Delta X]. \tag{9}$$

where $\mathbb{E}_k[H] = \mathbb{E}[H(\mu_k, \sigma_k)]$, $\mathbb{E}_k[\Delta X] = \mathbb{E}[\Delta X(\mu_k, \sigma_k)]$, $Cov_k(H, X_1) = Cov((H(\mu_k, \sigma_k), X_1(\mu_k, \sigma_k))$, for $k = 1, \ldots, r$.

Proof: Observe that the function $\mathbb{E}[g_\omega]$ is convex in V_0, ξ for fixed μ, σ. This is because the function g_ω viewed as a function of V_0, ξ for fixed μ, σ is a convex function jointly in V_0, ξ as it is the square of an affine expression in V_0, ξ and expectation over ω preserves convexity. Since pointwise supremum of a convex function is also a convex function, maximization over μ, σ yields a function convex in V_0, ξ. Thus, by Theorem 3.2 of [9] (Chapter 6, pp. 196–197), we have that a pair V_0^*, ξ^* is optimal in [RobQH2] if and only if there exist pairs $(\mu_1, \sigma_1) \ldots, (\mu_r, \sigma_r)$, $1 \leq r \leq 3$ such that

$$\max_{(\mu,\sigma)\in\mathcal{U}} \Phi(V_0, \xi, \mu, \sigma) = \Phi(V_0, \xi, \mu_k, \sigma_k),\ k = 1, \ldots, r$$

and r non-negative numbers $\alpha_k \geq 0$, $\sum_{k=1}^r \alpha_k = 1$ such that

$$\sum_{k=1}^r \alpha_k \frac{\partial \mathbb{E}_\omega[g_\omega(\mu_k, \sigma_k, V_0^*, \xi^*)]}{\partial V_0} = 0$$

and

$$\sum_{k=1}^r \alpha_k \frac{\partial \mathbb{E}_\omega[g_\omega(\mu_k, \sigma_k, V_0^*, \xi^*)]}{\partial \xi} = 0.$$

The result then follows by simple algebra. ∎

Based on this result, we can speak of the existence of a measure $\{\alpha_k\}_{k=1}^r$ such that

$$C_0 = \mathbb{E}^\alpha[\mathbb{E}[\{C_1\}_k]].$$

where $\mathbb{E}[\{C_1\}_k]$ is the expected value (with respect to P) of the cumulative cost of period 1, computed using (μ_k, σ_k) of Proposition 1, for $k = 1, \ldots, r$, $r \leq 3$.

4 Numerical Solution of Robust Problems

To be able to process any of the problems of the previous sections numerically we propose first to pose the problem [RobQH2] in variables V_0, ξ in the equivalent epigraph form in variables z, V_0, ξ

$$\begin{array}{ll} \min & z \\ \text{s.t.} & z \geq \mathbb{E}_\omega[g_\omega(\mu, \sigma, V_0, \xi)]\ \forall (\mu, \sigma) \in \mathcal{U} \end{array}$$

since all the problems treated in the paper can be viewed as a special case of [RobQH2]. From the proof of Proposition 1 we have that the constraints

$$z \geq \mathbb{E}_\omega[g_\omega(\mu, \sigma, V_0, \xi)]$$

for given $(\mu, \sigma) \in \mathcal{U}$ define each a convex set. Therefore, the epigraph form of [RobQH2] is an optimization problem with a linear objective function and a semi-infinite set of convex inequalities.

Now, we will be concerned with the following approach to solve problem [RobQH2]:

> Assume the support $\mathcal{U}$ for (μ, σ) is endowed with a σ-algebra $\mathcal{D}$ and a probability measure $\mathcal{P}$ over $\mathcal{D}$ is assigned. Given $\epsilon > 0$, replace the epigraph form of [RobQH2] with its *scenario counterpart* [SRobQH2]
>
> $$\begin{array}{ll} \min & z \\ \text{s.t.} & z \geq \mathbb{E}_\omega[g_\omega(\mu^j, \sigma^j, V_0, \xi)] \; j = 1, \ldots, N. \end{array}$$
>
> The question is: How large should the sample size N be in order for the optimal solution $V_{0,N}, \xi_N$ of the scenario problem to be feasible for the epigraph-form [RoBQH2] with probability at least $1 - \epsilon$?

The above question was answered in two recent contributions [5, 6]. To describe this approach, let us define the concept of *violation probability.* Let $z, V_0, \xi \in \mathbb{R}^3$ be given. Then the probability of violation of z, V_0, ξ is defined as

$$Vl(z, V_0, \xi) = \text{Prob}\{(\mu, \sigma) \in \mathcal{U} : z < \mathbb{E}_\omega[g_\omega(\mu, \sigma, V_0, \xi)]\}.$$

An ϵ-level feasible solution, for $\epsilon \in (0, 1)$, is defined as a triplet (z, V_0, ξ) such that $Vl(z, V_0, \xi) \leq \epsilon$. For convenience in the presentation, we assume that [SRobQH2] has a unique optimal solution. The following is a consequence of Theorem 1 of [6] in our context:

Proposition 2 *Fix two real numbers $\epsilon \in (0, 1)$ (level parameter) and $\beta \in (0, 1)$ (confidence parameter). If*

$$N \geq \frac{2}{\epsilon} \ln \frac{1}{\beta} + 6 + \frac{6}{\epsilon} \ln \frac{2}{\epsilon}$$

then, with probability no smaller than $1 - \beta$, the scenario problem [SRobQH2] gives an optimal solution which is an ϵ-level solution.

In the above result, the statement "with probability no smaller than $1 - \beta$" is to be interpreted as follows. This is the probability of extracting a "bad" sample $(\mu^{(1)}, \sigma^{(1)}), \ldots, (\mu^{(N)}, \sigma^{(N)})$ such that the scenario optimal solution $(z_N, V_{0,N}, \xi_N)$ does not meet the ϵ-level feasibility property. In other words, the statement of the Proposition 2 gives the following probabilistic guarantee at $(z_N, V_{0,N}, \xi_N)$:

$$\text{Prob}^N \left\{\text{Prob}\{(\mu, \sigma) \in \mathcal{U} : z_N < \mathbb{E}_\omega[g_\omega(\mu, \sigma, V_{0,N}, \xi_N)]\} \leq \epsilon\right\} \geq 1 - \beta.$$

where the notation $\text{Prob}^N (= P \times \ldots \times P, N \text{times})$ is the probability measure in the space $\mathcal{U}^N$ of the multi-sample extraction $(\mu^{(1)}, \sigma^{(1)}), \ldots, (\mu^{(N)}, \sigma^{(N)})$.

For the experimental results reported below, we face the problem of choosing appropriate ϵ and β. For the purposes of keeping run times and memory usage at a reasonable level (within a few minutes of CPU time) we settled for $\epsilon = 0.005$ and $\beta = 0.0001$, which results in a scenario sample of $N = 10,880$ according to Proposition 2. We evaluate the expectation in each constraint corresponding to a given μ, σ sample using the expressions derived in Lemma 1 and Lemma 2. When we increase the scenario sample size from 10,800 to 60,000 in steps of 10,000, we noticed that there is a change in the computed solution values only at the fourth digit after the decimal. Therefore, we deemed reasonably accurate the values obtained with a scenario sample of $10,880$.

5 Experimental Results

In generating all the results of this section, we compute the robust hedging policies by solving the scenario optimization problems [SRobQH2] using the modeling language GAMS [4] and the nonlinear optimization package CONOPT [8] with default tolerance settings. All tests were conducted on a personal computer endowed with an Intel Pentium IV CPU running at 3 GHz clock speed.

We begin our experimental analysis with problems [RobQHσ] and [RobQHeσ]. We report in Figure 1 below the result of robust quadratic hedging under the uncertainty model (4) with $\bar{\mu} = 0.2$, $\bar{\sigma} = 0.4$, and σ in the interval $[0.2, 0.6]$, for an out-of-the-money European Call with strike price $K = 11$ with the current price of the underlying $X_0 = 10$. To generate the curves in the figure, we proceed as follows. We generate randomly an iid sample of σ values with size equal to 1000, from the interval $[0.2, 0.6]$. For each value of σ we generate 5000 random asset prices according to model (4) and calculate the additional cumulative cost of hedging $(\eta_1 - \eta_0)$ and compute the average additional cumulative cost over the 5000 trials. This procedure is repeated for both the nominal quadratic hedging strategy given by (2) and (3), and the robust quadratic hedging strategy obtained by solving problem [RobQHσ] (or, [RobQHeσ]) using the scenario optimization approach. It is clear from Figure 1 that the average additional cumulative cost of the nominal strategy is always significantly higher than that of the robust hedging strategy (see also discussion below). The results are plotted in ascending sorted order. We follow this procedure in generating all the figures of this section with the appropriate changes (e.g., we generate (μ, σ) pairs from the uncertainty ellipsoid for Figures 4, 5 and 6). A result similar to Figure 1 is reported in Figure 2 for an out-of-the-money Call with $K = 11$, everything else being identical to the experiment of Figure 1 except that the price model (6) is used. A result similar to Figure 1 is reported in Figure 3 for an at-the-money Call with $K = 10$, everything else being identical to the experiment of Figure 1.

A-posteriori Assessments. Once a robust (scenario) solution $(V_{0,N}, \xi_N)$ has been obtained one can run an a-posteriori test of the level of feasibility (optimality) using Monte-Carlo techniques as suggested in [5]. To do this, a new batch of M independent random samples of $\sigma \in [0.2, 0.6]$ is generated, and the *empirical probability* of constraint violation, say $\tilde{V}l_M(V_{0,N}, \xi_N)$, is computed using the formula

$$\tilde{V}l_M(V_{0,N}, \xi_N) = \frac{1}{M} \sum_{i=1}^{M} \mathbf{1}(z < \mathbb{E}_\omega[g_\omega(\mu, \sigma, V_0, \xi)])$$

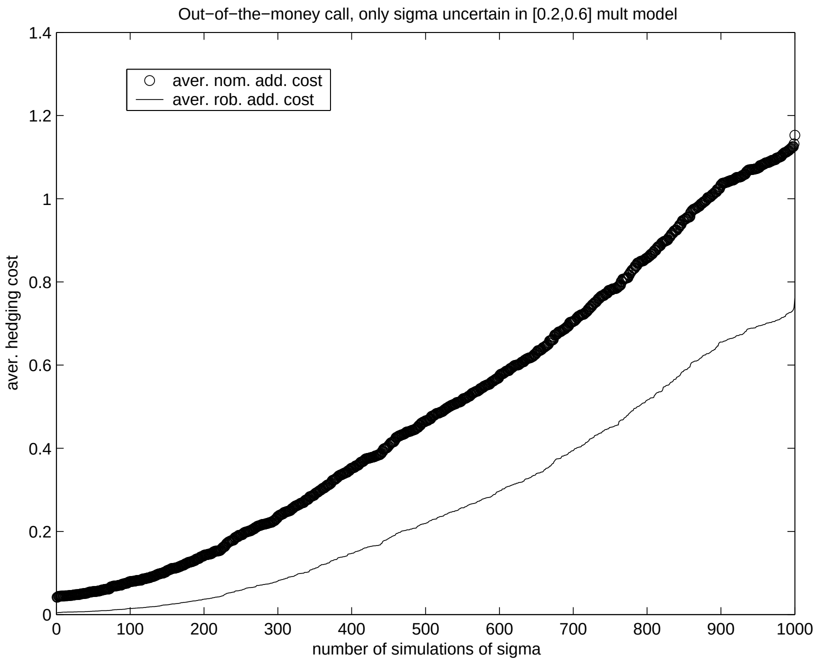

Figure 1: Out-of-the-money European Call: Average additional cost over 1000×5000 price movement simulations with $\bar{\mu} = 0.2,\ \bar{\sigma} = 0.4,\ X_0 = 10,\ K = 11$, and the price model (4)

where $\mathbf{1}(.)$ denotes the indicator function. Using Hoeffding's inequality, Calafiore and Campi [5] show that

$$\text{Prob}^M\{|\tilde{Vl}_M(V_{0,N}, \xi_N) - Vl(V_{0,N}, \xi_N)| \leq \tilde{\epsilon}\} \geq 1 - 2\mathrm{e}^{-2\tilde{\epsilon}^2 M}$$

from which it follows that $|\tilde{Vl}_M(V_{0,N}, \xi_N) - Vl(V_{0,N}, \xi_N)| \leq \tilde{\epsilon}$ holds with confidence greater than $1 - \tilde{\beta}$ if

$$M \geq \frac{\ln 2/\tilde{\beta}}{2\tilde{\epsilon}^2}.$$

For the experiment of Figure 1, we let $\tilde{\epsilon} = 0.001$, and $\tilde{\beta} = 0.00001$ (notice that these parameters are smaller than ϵ and β that we fixed for the optimization), from the above bound we obtain the lower bound on the Monte Carlo sample to be 6.103×10^6. Therefore, we have $|\tilde{Vl}_M(V_{0,N}, \xi_N) - Vl(V_{0,N}, \xi_N)| \leq 0.001$ with confidence greater that 99.999%. One run of the Monte Carlo yields

$$\tilde{Vl}(V_{0,N}, \xi_N) = 2.96 \times 10^{-5}.$$

This means that the hedging policy we have computed has violation probability of 0.0010296, i.e., it is optimal about 99.897% of the time.

For the experiments on RobQH2 we let the matrix $P = \begin{pmatrix} 0.1 & 0.01 \\ 0.01 & 0.1 \end{pmatrix}$ and thus allow joint uncertainty in the specification of both μ and σ. In Figure 4, we give results for an out-of-the-money Call with exactly the same characteristics as in the experiment of Figure 1 while we assume that μ, σ are allowed to vary within an ellipsoidal set defined by $\bar{\mu}, \bar{\sigma}$ and P. In Figure 5, we repeat this

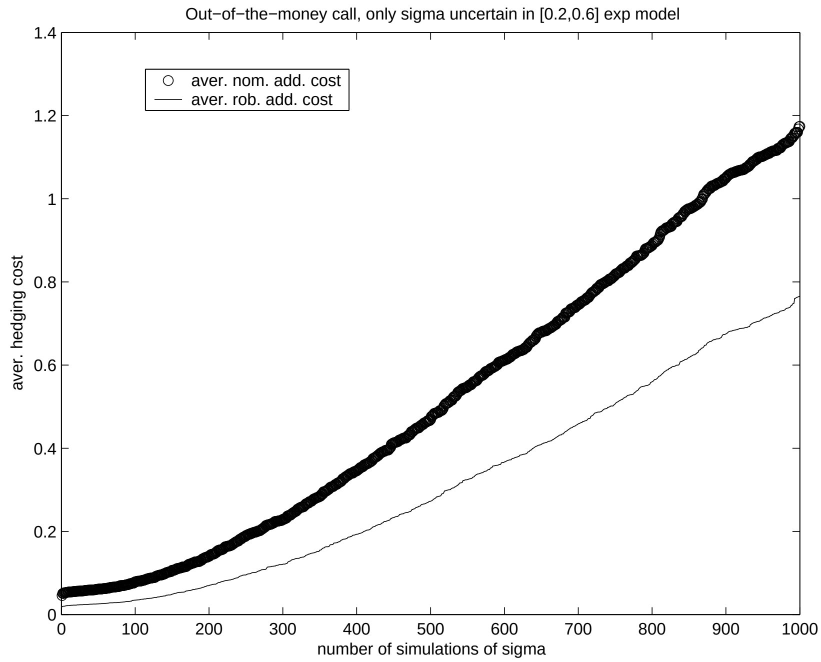

Figure 2: Out-of-the-money European Call: Average additional cost over 1000×5000 price movement simulations, only σ uncertain, with $\mu = 0.2$, $\bar{\sigma} = 0.4$, $X_0 = 10$, $K = 11$, and the price model (6)

experiment with an at-the-money Call as in Figure 3. Finally, Figure 6 gives results for an out-of-the-money Call with the same data as in the experiment of Figure 4, where both μ and σ are uncertain, using the asset price model (6).

It appears that in all the figures (experiments) the nominal policy results in larger additional cumulative hedging cost during simulations. An interpretation and assessment of this observation can be made as follows. In all the experimental results with price model (4), we observed that the robust hedging policy usually advocates taking a smaller long position ξ in the risky asset and a smaller short position in the riskless asset. This policy results in the value V_0 being larger in the robust policy than the value of V_0 in the nominal policy. For example, for the experiment of Figure 1, we obtain a robust policy of $\xi = 0.566$ and $\eta_0 = -3.867$ compared to 0.599 and -5.042, respectively, for the nominal policy. This corresponds to a $V_0 = 1.793$ for the robust policy, and to 0.948 for the nominal policy, an increase of %89 over the nominal policy if we think of V_0 as the cost of assembling a hedge portfolio at time $t = 0$. However, this increase can be more than compensated for in the second period. When we run 10 million simulations of the stock price X_1, we find that the extra additional cumulative cost $\eta_1 - \eta_0$ paid by the nominal policy over the additional cumulative cost paid by the robust policy (i.e., we take the difference $(\eta_1^{nom} - \eta_0^{nom}) - (\eta_1^{rob} - \eta_0^{rob})$) exceeds the difference $1.793 - 0.948$ in 3,943,747 cases in a single simulation run. Therefore, ignoring ambiguity may result in 39 percent chance of paying an additional cumulative hedging cost in time $t = 1$ larger than that would be paid by a robust policy.

In the results obtained with price model (6), the hedge portfolios are closer in values while the

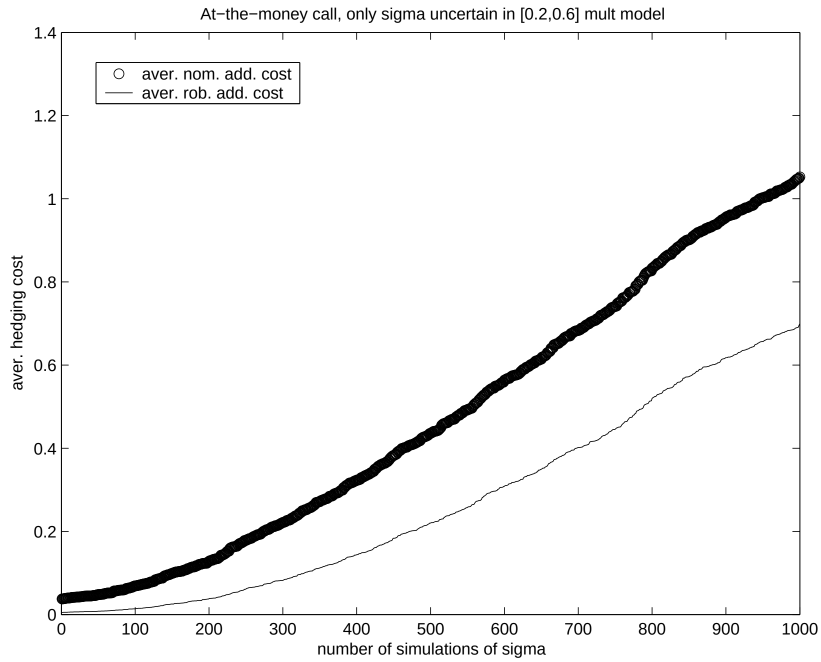

Figure 3: At-the-money European Call: Average additional cost over 1000×5000 price movement simulations with $\bar{\mu} = 0.2$, $\bar{\sigma} = 0.4$, $X_0 = 10$, $K = 10$, and the price model (4)

robust long position in the risky asset could be slightly larger than the nominal long position. For example, for the experiment of Figure 6, we obtain a robust policy of $\xi = 0.829$ and $\eta_0 = -7.365$ compared to 0.805 and -7.419, respectively, for the nominal policy. This corresponds to a $V_0 = 0.925$ for the robust policy, and to 0.631 for the nominal policy, an increase of $\%46$ over the nominal policy. However, this increase is again justified in the second period. When we run 10 million simulations of the stock price X_1, we find that the extra additional cumulative cost $\eta_1 - \eta_0$ paid by the nominal policy over the additional cumulative cost paid by the robust policy (as in the previous paragraph) exceeds the difference $0.925 - 0.631$ in 2,826,746 cases in a single simulation run. Therefore, ignoring ambiguity may result in roughly 28 percent chance of paying an additional cumulative hedging cost in time $t = 1$ larger than that would be paid by a robust policy in this example.

An application of robust optimization to multi-period hedging problems is left as a topic for future research.

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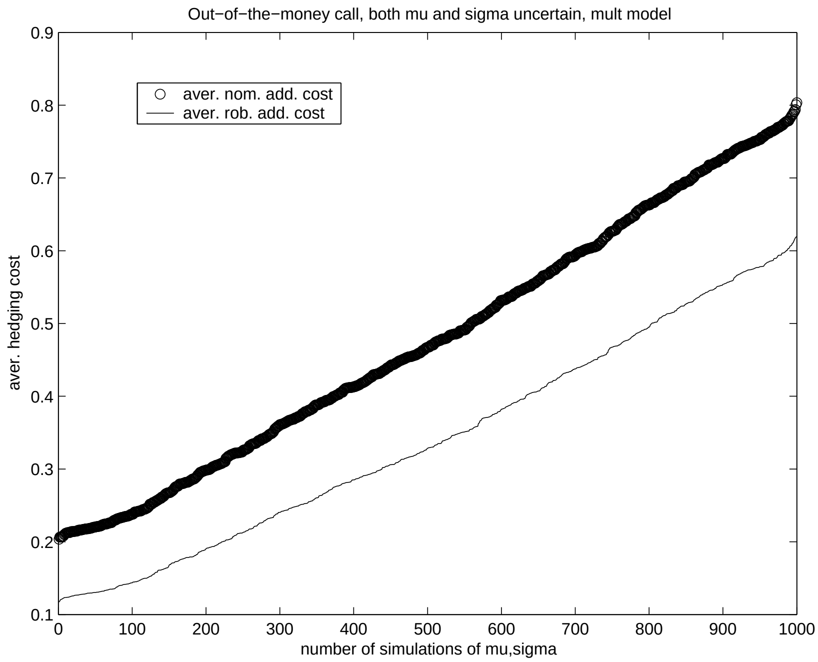

Figure 4: Out-of-the-money European Call: Average additional cost over 1000×5000 price movement simulations with $\bar{\mu} = 0.2$, $\bar{\sigma} = 0.4$, $X_0 = 10$, $K = 11$, and the price model (4)

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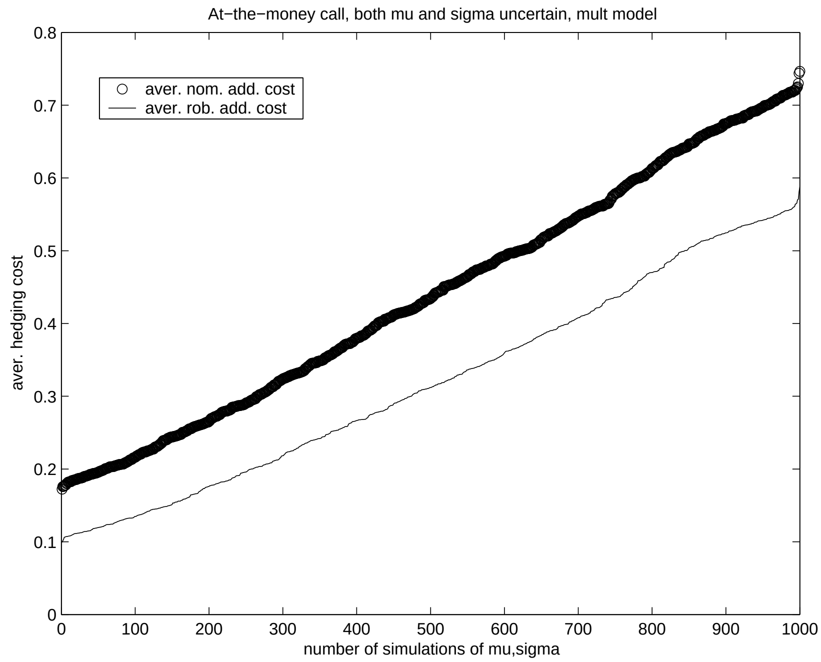

Figure 5: At-the-money European Call: Average additional cost over 1000×5000 price movement simulations, both μ and σ uncertain, with $\mu = 0.2$, $\bar{\sigma} = 0.4$, $X_0 = 10$, $K = 11$, and the price model (4)

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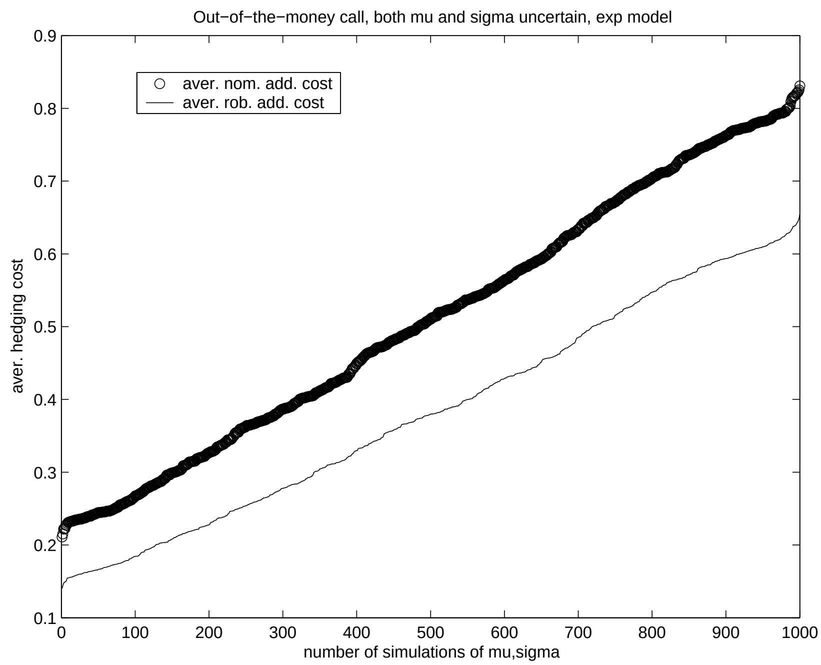

Figure 6: Out-of-the-money European Call: Average additional cost over 1000×5000 price movement simulations, with $\mu = 0.2$, $\bar{\sigma} = 0.4$, $X_0 = 10$, $K = 11$, and the price model (6)