Sharpe-Ratio Pricing and Hedging of Contingent Claims in Incomplete Markets by Convex Programming

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Abstract

We analyze the problem of pricing and hedging contingent claims in a financial market described by a multi-period, discrete-time, finite-state scenario tree using an arbitrage-adjusted Sharpe-ratio criterion. We show that the writer's and buyer's pricing problems are formulated as conic convex optimization problems which allow to pass to dual problems over martingale measures and yield tighter pricing intervals compared to the interval induced by the usual no-arbitrage price bounds. Furthermore, the bounds may collapse to a unique price at a limiting value of a risk aversion parameter, dictated by the financial market. This limiting value is computable as the optimal value of a convex quadratic program. An extension with proportional transaction costs for risky assets is given, as well as several numerical examples. Furthermore, to address the dependence of pricing results on the original probability measure used in describing the financial market, a model using several trial measures is developed.

Key words. Contingent claim, pricing, hedging, Sharpe ratio, martingales, convex programming.

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1 Introduction

A fundamental question of financial economics is to price an uncertain future stream of payments, i.e., a stochastic future cash flow. The prevailing approach to pricing is to replicate the uncertain cash flow using existing financial instruments and to find a price relative to these instruments so as to avoid an arbitrage opportunity in the financial market (hedging). When the financial market is complete, i.e., all stochastic future cash flows can be replicated using existing liquid instruments this approach yields unique sets of prices without any assumptions about individual investor's preferences. Ross [20, 21] proves that the no-arbitrage condition is equivalent to the existence of a linear pricing rule and positive state prices that correctly value all assets. This linear pricing rule comes from the risk neutral probability measure in the Cox-Ross option pricing model; for example Harrison and Kreps [12] showed that the linear pricing operator is an expectation taken with respect to a martingale measure.

The pricing problem is complicated by the fact that most financial markets are incomplete, i.e., not all future uncertain cash flows can be replicated exactly using the existing instruments. This observation leads to a wealth of literature on pricing and hedging in incomplete markets; see e.g., [3, 5, 8, 22]. When markets are incomplete state prices and claim prices are not unique. Since markets are almost never complete due to market imperfections as discussed in Carr *et al.* [5], and characterizing all possible future states of economy is impossible, alternative incomplete pricing theories have been developed. The common practice is to find the cheapest portfolio dominating a stochastic future cash flow and the most expensive portfolio dominated by it, and use these respective values as bounds on the price of the stochastic cash flow. These bounds are referred to as super-replication and sub-replication bounds or no-arbitrage (or equilibrium) bounds. We use these bounds as a benchmark in the present paper.

A second class of pricing theories relies on the Expected Utility Hypothesis and requires the specification of investor preferences. This model equates the price of a claim to the expectation of the product of the future payoff and the marginal rate of substitution of the representative investor. While this class of models has strong theoretical appeal (see e.g. [7, 14, 16] for related recent work), the need for specification of preferences, especially for the representative agent, limits their usefulness. Recent papers by Cochrane and Saa-Requejo [9], Bernardo and Ledoit [3], Carr *et al.* [5] and Roorda *et al.* [19] and Kallsen [16] unify these two classes of pricing theories and value options in an incomplete market setting.

Against this background, the purpose of this paper is to investigate how an arbitrage-adjusted Sharpe ratio criterion [6, 15] and convex optimization can help us develop a simple valuation framework in an incomplete market. All optimization models developed in the present paper are solvable by off-the-shelf optimization software. The arbitrage-adjusted Sharpe ratio remedies some shortcomings of the usual Sharpe ratio; see [6] for a discussion. Cerny [7] studies the use of the arbitrage-adjusted Sharpe ratio using some utility functions and essentially a dual approach in the space of pricing operators for pricing in incomplete markets.

In the present paper we show in a multi-period framework that in the absence of arbitrage (i.e., in the absence of infinite Sharpe ratios) while aiming for a finite Sharpe ratio and giving up a totally risk averse attitude in the non-negative terminal wealth positions, the buyer and the writer can agree on a common price to the contingent claim in an incomplete market. In the rest of this paper, we shall make these ideas precise. The present work is inspired by the contributions of Cochrane and Saa-Requejo [9] and those of Cerny [7, 6] where the authors develop similar ideas but essentially departing from the dual space of pricing measures. We start from the more natural primal hedging space which is the domain where investors operate rather than from the space of martingale measures. In the simple framework of discrete-time, multiperiod, finite state probability framework, we show how the problems of writers and buyers are formulated as convex (conic) optimization problems, and pass to the results of [9, 7] using convex programming duality. Furthermore, we show that the limiting value of the arbitrage-adjusted Sharpe ratio leading potentially to a unique price of a contingent claim can be computed as the optimal value of a convex quadratic programming problem. We illustrate our results with numerical examples throughout the text. The present paper can be seen as a sequel or companion to our previous work [18] where pricing in incomplete markets was studied using the

Bernardo-Ledoit [3] approach based on the ratio of positive expected final wealth to negative final wealth positions, and linear programming.

The rest of this paper is organized as follows. In section 2 we describe the setting for our financial market and review the fundamental theorem of asset pricing. We also formulate in a multi-period framework the buyer and writer no-arbitrage pricing problems along with their dual representations in terms of optimization over martingale measures. In section 3 we formulate the pricing problems with an arbitrage-adjusted Sharpe ratio restriction and describe the dual problems. Section 4 studies the pricing problem after including proportional transaction costs for buying and selling risky instruments. In section 5 we consider several measures describing the evolution of financial market and enforce the arbitrage-adjusted Sharpe ratio criterion for all these measures simultaneously to alleviate the dependence of the pricing results on a single measure.

2 The Stochastic Scenario Tree, Arbitrage, Martingales and Pricing Bounds

Throughout this paper we follow the general probabilistic setting of [17] in that we approximate the behavior of the stock market by assuming that security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms are sequences of real-valued vectors (asset values) over the discrete time periods $t = 0, 1, \ldots, T$. We further assume the market evolves as a discrete, non-recombinant scenario tree (hence, suitable for incomplete markets) in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted $\pi(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$ has a non-empty set of child nodes $\mathcal{S}(n) \subset \mathcal{N}_{t+1}$. We denote the set of all nodes in the tree by $\mathcal{N}$. The probability distribution P is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-terminal (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m \in \mathcal{S}(n)} p_m, \;\; \forall\, n \in \mathcal{N}_t, \; t = T-1, \ldots, 0.$$

Hence, each intermediate node has a probability mass equal to the combined mass of the paths passing through it. The ratios $p_m/p_n, m \in \mathcal{S}(n)$ are the conditional probabilities that the child node m is visited given that the parent node $n = \pi(m)$ has been visited.

A random variable X is a real valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω, [17]. This kind of random variable is said to be measurable with respect to the information contained in the nodes of $\mathcal{N}_t$. A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is measurable with respect $\mathcal{N}_t$. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^P[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is given by the expression

$$\mathbb{E}^P[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{S}(n)} \frac{p_m}{p_n} X_m.$$

Note that this conditional expectation is a random variable taking values of the nodes $n \in \mathcal{N}_t$. Under the light of the above definitions, the market consists of $J+1$ tradable securities indexed by $j = 0, 1, \ldots, J$ with prices at node n given by the vector $S_n = (S_n^0, S_n^1, \ldots, S_n^J)$. We assume as in [17] that the security indexed by 0 has strictly positive prices at each node of the scenario tree. This asset corresponds to the risk-free asset in the classical valuation framework. Choosing this security as the numéraire, and using the discount factors $\beta_n = 1/S_n^0$ we define $Z_n^j = \beta_n S_n^j$ for $j = 0, 1, \ldots, J$ and $n \in \mathcal{N}$, the security prices discounted with respect to the numéraire. Note that $Z_n^0 = 1$ for all nodes $n \in \mathcal{N}$.

The amount of security j held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted θ_n^j. Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$. We refer to the collection of vectors $\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|}$

as Θ. The value of the portfolio at state n (discounted with respect to the numéraire) is

$$Z_n \cdot \theta_n = \sum_{j=0}^{J} Z_n^j \theta_n^j.$$

We will work with the following definition of arbitrage: an arbitrage is a sequence of portfolio holdings that begins with a zero initial value (note that short sales are allowed), makes self-financing portfolio transactions throughout the planning horizon and achieves a non-negative terminal value in each state, while in at least one terminal state it achieves a positive value with non-zero probability. The self-financing transactions condition is expressed as

$$Z_n \cdot \theta_n = Z_n \cdot \theta_{\pi(n)}, \; n > 0.$$

The stochastic programming problem used to seek an arbitrage is the following optimization problem (P1):

$$\begin{array}{lrcl} \max & \sum_{n \in \mathcal{N}_T} p_n Z_n \cdot \theta_n & & \\ \text{s.t.} & Z_0 \cdot \theta_0 & = & 0 \\ & Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & 0, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n & \geq & 0, \; \forall\, n \in \mathcal{N}_T. \end{array}$$

If there exists an optimal solution (i.e., a sequence of $\theta_0, \theta_1, \ldots, \theta_{|\mathcal{N}|}$ vectors) which achieves a positive optimal value, this solution can be turned into an arbitrage as demonstrated by Harrison and Pliska [13].

We need the following definitions.

Definition 1 *If there exists a probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ such that*

$$Z_t = \mathbb{E}^Q[Z_{t+1} | \mathcal{N}_t] \; (t \leq T-1) \tag{1}$$

then the vector process $\{Z_t\}$ is called a vector-valued martingale under Q, and Q is called a martingale probability measure for the process.

Definition 2 *A discrete probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ is equivalent to a (discrete) probability measure $P = \{p_n\}_{n \in \mathcal{N}_T}$ if $q_n > 0$ exactly when $p_n > 0$.*

King proved the following (c.f. Theorem 1 of [17]) result that is a re-statement of Theorem 1 of Harrison and Kreps [12] in our setting (see also [20, 21]):

Theorem 1 *The discrete state stochastic vector process $\{Z_t\}$ is an-arbitrage free market price process if and only if there is at least one probability measure Q equivalent to P under which $\{Z_t\}$ is a martingale.*

Example 1. Consider a single period financial market example taken from [11], composed of a riskless asset and a stock. For simplicity, assume that the riskless asset does not pay interest, and that its current price is equal to 1 (this implies $\beta_0 = 1$. Furthermore, the stock price which is currently equal to 10 can become 20, or 15, or 7.5 with equal probability in the next period. By our assumption, the riskless asset price remains equal to 1 in the next time period (all β_n for $n > 0$ are equal to one). One can think of this price process as a two-period scenario tree where in the first period there is the single node 0 corresponding to the initial price equal to 10, and in the second period each possible price gives rise to a node in the scenario three, resulting in node 1 (stock price equal to 20), node 2 (stock price equal to 10) and finally node 3 (stock price equal to 7.5). Following the theory of financial markets [11], this is a simple example of an incomplete market where all stochastic future pay-offs may not be exactly replicated by forming a portfolio composed of the riskless asset and the stock with the aim of avoiding an arbitrage. It turns out that the incomplete market described above is arbitrage free since one can exhibit a martingale measure equivalent to the original measure that makes the stock price process a martingale, e.g., $q_1 = 0.136$, $q_2 = 0.107$, $q_3 = 0.757$.

Consider now the pricing problem of a contingent claim F with pay-offs $F_n, n > 0$. The seller of the contingent claim is interested in solving the following problem (WA)

$$\begin{aligned} \min \quad & Z_0 \cdot \theta_0 \\ \text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) \;=\; -\beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & 0 \;\leq\; Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \end{aligned}$$

while the buyer's problem of finding the maximum acceptable price would be (BA)

$$\begin{aligned} \max \quad & -Z_0 \cdot \theta_0 \\ \text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) \;=\; \beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & \qquad\qquad 0 \;\leq\; Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T. \end{aligned}$$

The writer's problem deals with the following: what is the actual (at the current prices) cost of a self-financed portfolio process that replicates the pay-outs that will be due to the holder of the claim without risking negative terminal wealth positions in any of the states of the market? Among such portfolios, one giving the smallest cost portfolio is an optimal portfolio, and its cost is the minimum amount the writer should charge to sell the claim. The buyer's problem is simply the opposite. What is the maximum he/she should be prepared to pay to acquire a particular claim? At most the current value of the optimal portfolio that replicates the proceeds from the claim in a self-financed manner and at no risk of negative terminal positions. We refer to the optimal values of the above problems commonly as "no-arbitrage bounds".

Under the assumption of no-arbitrage in the market, these problems lead to the following pricing interval obtained using linear programming duality; see [17] for details:

$$[\frac{1}{\beta_0} \min_{q \in \mathcal{Q}} \mathbb{E}^Q [\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{q \in \mathcal{Q}} \mathbb{E}^Q [\sum_{t=1}^{T} \beta_t F_t]]$$

where Q represents the set of all measures that make the price process a martingale (not necessarily equivalent to P).

Example 2. Continuing with our example of incomplete market, let us assume that a call option of the European type that matures in the next time period is written on the stock. The exercise price (strike) of the option is equal to 9. The option introduces to the market a stochastic future pay-off contingent on the state of the stock price. In particular, the option pays 11 to the holder in case the stock price becomes 20, 6 in case the stock price is 15, and nothing if the stock price is 7.5. It is easy to see that no portfolio consisting of the riskless asset and the stock can exactly replicate the future pay-off stream. Hence, one can resort to the above optimization problems WA and BA to compute writer and buyer no-arbitrage bounds for this option. Let B_0 and S_0 denote the amount of the riskless asset and of the stock in the portfolio at time $t = 0$, respectively, or at node 0 of the scenario tree. The holdings in the riskless asset and the stock are denoted respectively B_1, S_1, B_2, S_2 and B_3, S_3 for the nodes 1, 2 and 3 of scenario tree at $t = 1$. (We avoid the use of θ_n in the examples to avoid unnecessary clutter). The writer wishes to choose B_0 and S_0 so as to make $B_0 + 10S_0$ as small as possible while he/she wants to cover the pay-out originating from the claim at nodes 1, 2 and 3 by self-financing transactions, i.e., changes to B_0 and S_0 involving no inflow not outflow of cash such that the final wealth at nodes 1, 2 and 3 are all non-negative. The buyer's problem is exactly the opposite. The seller's and buyer's pricing problems are then the following problems:

$$\min B_0 + 10S_0$$

subject to

$$B_1 - B_0 + 20(S_1 - S_0) = -11$$

$$B_2 - B_0 + 15(S_2 - S_0) = -6$$

$$B_3 - B_0 + 7.5(S_3 - S_0) = 0$$

$$B_1 + 20S_1 \geq 0$$

$$B_2 + 15S_2 \geq 0$$

$$B_3 + 7.5S_3 \geq 0$$

with the dual problem

$$\max_{(q_1,q_2,q_3)\geq 0} 11q_1 + 6q_2$$

subject to

$$q_1 + q_2 + q_3 = 1$$

$$20q_1 + 15q_2 + 7.5q_3 = 10,$$

for the seller. The corresponding problems for the buyer are

$$\max -B_0 - 10S_0$$

subject to

$$B_1 - B_0 + 20(S_1 - S_0) = 11$$

$$B_2 - B_0 + 15(S_2 - S_0) = 6$$

$$B_3 - B_0 + 7.5(S_3 - S_0) = 0$$

$$B_1 + 20S_1 \geq 0$$

$$B_2 + 15S_2 \geq 0$$

$$B_3 + 7.5S_3 \geq 0,$$

$$\min_{(q_1,q_2,q_3)\geq 0} 11q_1 + 6q_2$$

subject to

$$q_1 + q_2 + q_3 = 1$$

$$20q_1 + 15q_2 + 7.5q_3 = 10,$$

Following [17] we know that these problems are solvable since the market is arbitrage free. The optimal solutions) turn out to be equal to 2 (buyer's optimal value) and 2.2 (writer's optimal value) in this case. The first quantity is the largest amount a potential buyer would pay to acquire the call option while the second is the minimum amount a potential seller (writer) of the option is willing to charge a buyer to be willing to sell the option. The hedging strategies of the buyer and of the writer, are as follows, respectively. The buyer forms a portfolio consisting of 6 units of the riskless asset and shorts 0.8 units of the stock. At the next period, if the stock price evolves to 20, the buyer passes to a position of 1 unit in the riskless asset, and zeroes out its position in the stock. If the stock price evolves to 15 or to 7.5, the buyer zeroes out all its positions. The writer on the other hand forms a portfolio composed of 0.8 units of the stock and shorts -6.6 units of the riskless asset. In the second and final time period, if the stock price evolves to 15, the writer passes to a long position of 0.6 units in the riskless asset, and zeroes out holdings in the stock. In the other cases, all positions are zeroed out. The dual problem for the buyer has the optimal solution $q_1 = 0$, $q_2 = 1/3$, and $q_3 = 2/3$. The optimal solution for the writer's dual problem is $q_1 = 1/5$, $q_2 = 0$, $q_3 = 4/5$.

For any price for the option below 2, it can be shown that the buyer can make an arbitrage profit, and for any price above 2.2, the seller can realize an arbitrage profit. Since no trading activity should take place at any price between 2 and 2.2 according to the above analysis, this situation leaves one with the intriguing question of how to put a single price tag on this option, which seems to be impossible with the approach outlined so far.

3 Pricing Bounds using the Arbitrage-Adjusted Sharpe Ratio

In this section we show in a multi-period framework that in the absence of arbitrage (i.e., in the absence of infinite Sharpe ratios) while aiming for a finite Sharpe ratio reachable by giving up a totally risk averse attitude in the non-negative terminal wealth positions, the buyer and the writer can agree on a common price to the contingent claim in an incomplete market. To this end we adopt the approach of [7, 15] where the set of investment opportunities which are not necessarily arbitrage opportunities but which may be "good-deals", is represented as the convex hull of investments with non-negative outcomes and those with high Sharpe ratio. This representation consists in expressing the outcome of an investment as a random variable with two components: a non-negative component tracking an arbitrage position, and an unrestricted-in-sign component tracking a high Sharpe ratio. Let the random variable X represent the uncertain cash flow of an investment. The random variable X is split into two components:

$$X = X_{sh} + X_{arb},$$

where X_{arb} is a non-negative random variable measuring the arbitrage (zero risk) component of the cash flow (we will illustrate this concept in the sequel), and X_{sh} measures the Sharpe ratio component of X. Then the arbitrage-adjusted Sharpe ratio of X is defined as

$$\frac{\mathbb{E}[X]}{\sqrt{\sigma(X)}}$$

where $\sigma(X)$ denotes the variance of the random variable X. We adapt this formulation to our setting. The writer is concerned with the following question: what is the actual (at the current prices) cost of a self-financed portfolio process that replicates the pay-outs that will be due to the holder of the claim while risking some negative terminal wealth positions held in check by a restriction on the Sharpe ratio? Among such portfolios, one giving the smallest cost portfolio is an optimal portfolio, and its cost is the minimum amount the writer should charge to sell the claim. The buyer's problem is simply the opposite. What is the maximum he/she should be prepared to pay to acquire a particular claim? At most the current value of the optimal portfolio that replicates the proceeds from the claim in a self-financed manner and allowing some negative terminal positions controlled by a restriction on Sharpe ratio.

The writer's problem for financing a contingent claim F is (referred to as WSP)

$$\begin{aligned} \min \quad & Z_0 \cdot \theta_0 \\ \text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & \sum_{n\in\mathcal{N}_T} p_n x_n - \lambda \sqrt{\sum_{n\in\mathcal{N}_T} p_n (x_n - \sum_{n\in\mathcal{N}_T} p_n x_n)^2} \geq 0 \\ & v_n + x_n = Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\ & v_n \geq 0 \ \forall\, n \in \mathcal{N}_T, \end{aligned}$$

while the buyer's problem of finding the maximum acceptable price is (referred to as BSP)

$$\begin{aligned} \max \quad & -Z_0 \cdot \theta_0 \\ \text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = \beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & \sum_{n\in\mathcal{N}_T} p_n x_n - \lambda \sqrt{\sum_{n\in\mathcal{N}_T} p_n (x_n - \sum_{n\in\mathcal{N}_T} p_n x_n)^2} \geq 0, \\ & v_n + x_n = Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\ & v_n \geq 0 \ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

The arbitrage-adjusted Sharpe ratio of terminal wealth positions is enforced through a positive parameter λ. Notice that when λ increases to ∞, one recovers the problems WA and BA of the previous section, respectively, after suppressing the variables x_n, i.e., we reach the arbitrage pricing theory in the limit as λ increases without bound.

Notice that both problems above are convex optimization (in fact, conic) problems. We analyze first the writer's problem using Lagrange duality since the analysis of the buyer's problem is similar. Forming the Lagrange function and carrying out the separate maximizations over the variables x_n, v_n, Θ, respectively, we obtain the following dual problem in the variables $\omega = \omega_{n,n\in\mathcal{N}_T}$, $\{q_n\}_{n\in\mathcal{N}_T}$

$$\max \sum_{n>0} \beta_n q_n F_n \tag{2}$$

subject to

$$\|\omega\|_2 \leq \lambda, \tag{3}$$

$$q_n = p_n(1 + \tilde{p}^T\omega) - \sqrt{p_n}\omega_n, \forall n \in \mathcal{N}_T, \tag{4}$$

$$q_m Z_m = \sum_{n\in\mathcal{S}(m)} q_n Z_n, \ \forall\ m \in \mathcal{N}_t, 0 \leq t \leq T-1 \tag{5}$$

$$q_0 = 1, \tag{6}$$

$$q_n \geq 0, \ \forall n \in \mathcal{N}_T, \tag{7}$$

where $\tilde{p}$ is the vector with components $\sqrt{p_n}$. More details on the derivation of this dual problem are given in the Appendix. The dual of the buyer's problem is simply

$$\min \sum_{n>0} \beta_n q_n F_n \tag{8}$$

subject to (3)–(7), i.e., minimizing the same objective over the same set of constraints. Using the property that the first entry of Z_n is equal to 1 for all n, we have that the vector q_n is a probability measure that makes the price process $\{Z_t\}$ a martingale. Now, observe that one can solve the equations (4) for ω_n, $n \in \mathcal{N}_T$, and obtain the following solution

$$\omega_n = -\sqrt{p_n}(q_n/p_n - 1), \ n \in \mathcal{N}_T.$$

Hence, it is immediate to see after elimination of ω_n's that the constraint (3) can be re-written as

$$\sqrt{\sigma(q./p)} \leq \lambda \tag{9}$$

with

$$\mathbb{E}[q./p] = 1, \tag{10}$$

where σ is the variance (with respect to measure P)

$$\sum_{n\in\mathcal{N}_T} p_n(\frac{q_n}{p_n} - \sum_{n\in\mathcal{N}_T} p_n\frac{q_n}{p_n})^2 = \sum_{n\in\mathcal{N}_T} p_n(\frac{q_n}{p_n} - 1)^2,$$

and $q./p$ denotes the vector with components q_n/p_n, for all $n \in \mathcal{N}_T$ (in fact, this is the Radon-Nikodym derivative $\frac{dQ}{dP}$). Therefore, we can state the following result whose proof follows from the previous development and the Conic Duality Theorem; see [1].

Theorem 2 *Assuming that problems BSP and WSP possess strictly feasible solutions*[1] *we have*

1. *the minimum price charged by a writer of contingent claim F desiring an arbitrage-adjusted Sharpe ratio of terminal wealth positions equal to λ or higher is given by*

$$\frac{1}{\beta_0} \max_{q\in\mathcal{Q}(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t]$$

   where the set $\mathcal{Q}$ is the set of all martingale measures for the discrete price process $\{Z_t\}$ satisfying the side conditions (9)-(10),

2. *the maximum price acceptable to the buyer seeking at least an arbitrage-adjusted Sharpe ratio of λ from the terminal wealth positions is given by*

$$\frac{1}{\beta_0} \min_{q\in\mathcal{Q}(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t].$$

Therefore, we obtain the following pricing interval whose end-points correspond to the maximum price a potential buyer is willing to pay to acquire the contingent claim and the minimum price the writer of the contingent claim is willing to charge, and all this while limiting the risk of falling short in the sense of a restriction on the arbitrage-adjusted Sharpe ratio of terminal wealth at level λ:

$$[\frac{1}{\beta_0} \min_{q\in\mathcal{Q}(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{q\in\mathcal{Q}(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t]].$$

[1] Conic (second-order cone) duality theory has some features that can lead to pathological cases of primal-dual problems. Since these cases are not essential to our development, we leave them aside, and refer the interested reader to the excellent book by Ben-Tal and Nemirosvki [1] for an extensive treatment of conic duality theory. The assumption of strict feasibility helps us to avoid these cases.

Notice that the above interval is a narrower interval in width compared to the arbitrage-free pricing interval as a result of the observation that both the writer and the buyer's dual problems have feasible sets contained in the feasible set of the arbitrage-bound problems of the previous section.

As λ is decreased, the dual feasible set shrinks since we have $\mathcal{Q}(\lambda_1) \subseteq \mathcal{Q}(\lambda_2)$ for $\lambda_1 \leq \lambda_2$. Therefore, we expect to be able to decrease λ to a limiting value beyond which the feasible set $\mathcal{Q}(\lambda)$ is empty. To compute this limiting point, one can simply solve the conic optimization problem referred to as ML (where λ is now a variable)

$$\min_{q,\lambda}\{\lambda : (4) - (5) - (6) - (7) - (9) - (10)\}.$$

Notice that the problem ML can be solved as a strictly convex quadratic programming problem after eliminating the variable λ and squaring the objective function:

$$\min_{q}\{\sigma(q./p) : (4) - (5) - (6) - (7) - (10)\}.$$

The above problem has unique optimal solution due to strict convexity of the objective function.

Example 3. Let us now return to the simple problem of Example 1 and Example 2, and examine the arbitrage-adjusted Sharpe ratio corresponding to the no-arbitrage pricing terminal wealth positions for the buyer and the writer. In both agents' hedging strategies all terminal positions are non-negative, therefore the Sharpe ratio components are all equal to zero, which implies that the respective terminal wealth positions lead to an infinite arbitrage-adjusted Sharpe ratio. Since investments with higher Sharpe ratios are considered preferable to those with lower Sharpe ratios, it appears here that the zero risk restriction (infinite arbitrage-adjusted Sharpe ratio) on the part of the buyer and of the writer puts a too stringent requirement for the prices to be meaningful. Settling for a smaller arbitrage-adjusted Sharpe ratio, the buyer and seller can form portfolios with limited terminal risk, but leading to eventually a unique price for the contingent claim.

Let us now give up on the requirement of non-negative terminal wealth positions, and aim for a finite arbitrage-adjusted Sharpe ratio (we will say simply Sharpe ratio for short) introducing a lower restriction on the Sharpe ratio of the terminal wealth positions as we have advocated in this section.

Therefore, we formulate the following pricing problems:

$$\min B_0 + 10S_0$$

subject to

$$B_1 - B_0 + 20(S_1 - S_0) = -11$$

$$B_2 - B_0 + 15(S_2 - S_0) = -6$$

$$B_3 - B_0 + 7.5(S_3 - S_0) = 0$$

$$B_1 + 20S_1 = x_1 + v_1$$

$$B_2 + 15S_2 = x_2 + v_2$$

$$B_3 + 7.5S_3 = x_3 + v_3$$

$$v_1 \geq 0, v_2 \geq 0, v_3 \geq 0$$

$$\frac{1}{3}(x_1+x_2+x_3) \geq \lambda\sqrt{\frac{1}{3}\left((x_1 - \frac{1}{3}(x_1+x_2+x_3))^2 + (x_2 - \frac{1}{3}(x_1+x_2+x_3))^2 + (x_3 - \frac{1}{3}(x_1+x_2+x_3))^2\right)}$$

with the dual problem

$$\max_{(q_1,q_2,q_3)\geq 0} 11q_1 + 6q_2$$

subject to

$$q_1 + q_2 + q_3 = 1$$

$$20q_1 + 15q_2 + 7.5q_3 = 10,$$

$$\sqrt{1/3((3q_1-1)^2 + (3q_2-1)^2 + (3q_3-1)^2)} \leq \lambda$$

for the seller. For the buyer we have

$$\max -B_0 - 10S_0$$

subject to

$$B_1 - B_0 + 20(S_1 - S_0) = 11$$

$$B_2 - B_0 + 15(S_2 - S_0) = 6$$

$$B_3 - B_0 + 7.5(S_3 - S_0) = 0$$

$$B_1 + 20S_1 = x_1 + v_1$$

$$B_2 + 15S_2 = x_2 + v_2$$

$$B_3 + 7.5S_3 = x_3 + v_3$$

$$v_1 \geq 0, v_2 \geq 0, v_3 \geq 0$$

$$\frac{1}{3}(x_1+x_2+x_3) \geq \lambda\sqrt{\frac{1}{3}\left((x_1 - \frac{1}{3}(x_1 + x_2 + x_3))^2 + (x_2 - \frac{1}{3}(x_1 + x_2 + x_3))^2 + (x_3 - \frac{1}{3}(x_1 + x_2 + x_3))^2\right)}$$

with the dual problem

$$\min_{(q_1,q_2,q_3)\geq 0} 11q_1 + 6q_2$$

subject to

$$q_1 + q_2 + q_3 = 1$$

$$20q_1 + 15q_2 + 7.5q_3 = 10,$$

$$\sqrt{1/3((3q_1 - 1)^2 + (3q_2 - 1)^2 + (3q_3 - 1)^2)} \leq \lambda.$$

By solving problem ML we find the smallest allowable value of λ equal to 0.8111 with an optimal solution $q_1 = 0.026$, $q_2 = 0.289$, and $q_3 = 0.684$. The writer's pricing bounds obtained by solving problem WSP for decreasing values of λ give optimal values that decrease toward a limiting value of 2.0263 from 2.2. For $\lambda = 1$, the bound is equal to 2.19, for $\lambda = 0.95$, it is equal to 2.165, for $\lambda = 0.9$, 2.135 and so on. For the buyer bounds, and the above values of λ (i.e., 1, 0.95 and 0.9) and until λ gets very close to the limiting value, we observe an optimal value of 2. For $\lambda = 0.815$ we observe an optimal value equal to 2.003 after which we obtain convergence to the same value as that of the writer. That is to say, for arbitrage-adjusted Sharpe ratio equal to 0.8111, both the writer and buyer pricing problems return the same optimal value which is equal to 2.0263, but achieve this value using quite different hedge portfolio strategies. In particular, the buyer commits to a portfolio that contains initially 95.501 units of stock, and shorts 957.038 units of the riskless asset. In the second time period, in case the stock price becomes 20, it zeroes out the short position in the riskless asset, and passes to a long position of 48.199 units in the stock. For the case where the asset price is 15 and 7.5, respectively, again the short position in the riskless asset is zeroed out, and the buyer passes to a long position of 32.099 in the stock in the former case, and to a short position of 32.104 in the stock in the latter. These correspond to a terminal wealth vector equal to $(963.956, 481.48, -240.779)$ with a Sharpe ratio equal to 0.8111 which is the lower bound specified.

The writer, on the other hand, forms initially a portfolio with 124.25 units of stock while shorting 1240.475 units of the riskless asset. In the second period, in all states she/he revises this portfolio by zeroing out the short position in the riskless asset, however passes to long positions of 61.676 and 41.152 units in the stock, when the asset price is 20 and 15, respectively. In case where the asset price is 7.5 she/he takes a short position of 41.147 units in the stock. This portfolio strategy yields the following terminal positions $(1233.528, 617.277, -308.59)$ with a Sharpe ratio equal to 0.8111.

Repeating this exercise in the above financial market for a European put option with strike $K = 14$, the common price equal to 4.447 is attained at the limiting value of $\lambda = 0.8111$, whereas the optimal buyer no-arbitrage price from problem BA is equal to 4.333 and the optimal writer no-arbitrage price from problem WA is equal to 5.2.

Example 4. Let us consider a two-period example. The financial market consists of a riskless asset whose price remains unity throughout, and of a risky stock valued at a price of 10 at $t = 0$, with the price evolving as follows. In time $t = 1$, the price can be 20, or 15 or 7.5 with equal probability as in the previous example. If the price is equal to 20 at time $t = 1$, it can become either 22, or 21 or 19 at $t = 2$ with equal probability. If it is equal to 15 at $t = 1$, it can change to 17 or 14 or 13 with equal probability. Finally, if it is equal to 7.5 at $t = 1$, it can evolve to 9 or 8 or 7 with equal probability at time $t = 2$. This financial market is arbitrage-free.

Now, let us introduce a European call option that matures at time $t = 2$ with strike price equal to 14. The pricing bounds obtained by solving problems BA and WA are 0.333 and 1.2, respectively. Formulating the problems WSP and BSP, we obtain the following price bounds for this call option given in Table 1. At the value of $\lambda = 1.086$, the two bounds coincide at the price 0.423 obtained using the martingale measure $q_4 = q_5 = q_6 = 0$, $q_7 = 0.141$, $q_8 = 0.101$, $q_9 = 0.091$, $q_{10} = 0.056$, $q_{11} = 0.222$, and $q_{12} = 0.389$.

λ	**BSP**	**WSP**
1.09	0.405	0.496
1.089	0.407	0.483
1.088	0.41	0.469
1.087	0.414	0.45
1.0865	0.4168	0.4387

Table 1: Arbitrage-adjusted Sharpe-ratio-based bounds for Example 4.

4 Proportional Transaction Costs

In financial markets the investor typically has to pay transaction costs proportional to the magnitude of the trade in risky assets; see [10] for a discussion. In this section we show how the approach of the previous section can be appropriately modified to take this aspect of financial markets into account.

Let $\bar{\theta}_n$ represent the sub-vector with m components of the vector θ_n, obtained by excluding the first component corresponding to the riskless asset. Let $\bar{Z}_n$ have an identical definition. We denote the transaction cost factor by η. Now, the writer's problem for financing a contingent claim F under a restriction λ on the arbitrage-adjusted Sharpe ratio is (referred to as WSPTC)

$$\begin{aligned}
\min \quad & Z_0 \cdot \theta_0 + \eta|\bar{Z}_0 \cdot \bar{\theta}_0| \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) + \eta|\bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)})| = -\beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \sum_{n\in\mathcal{N}_T} p_n x_n - \lambda \sqrt{\sum_{n\in\mathcal{N}_T} p_n (x_n - \sum_{n\in\mathcal{N}_T} p_n x_n)^2} \geq 0 \\
& v_n + x_n = Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\
& v_n \geq 0 \;\forall\, n \in \mathcal{N}_T,
\end{aligned}$$

while the buyer's problem of finding the maximum acceptable price is (referred to as BSPTC)

$$
\begin{aligned}
\max \quad & -Z_0 \cdot \theta_0 - \eta|\bar{Z}_0 \cdot \bar{\theta}_0| \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) + \eta|\bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)})| \;=\; \beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \sum_{n \in \mathcal{N}_T} p_n x_n - \lambda \sqrt{\sum_{n \in \mathcal{N}_T} p_n (x_n - \sum_{n \in \mathcal{N}_T} p_n x_n)^2} \;\geq\; 0 \\
& v_n + x_n \;=\; Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\
& v_n \;\geq\; 0 \;\forall\, n \in \mathcal{N}_T.
\end{aligned}
$$

We transform the above problems into equivalent problems without using the absolute value operator, and by a well-known trick of reformulation that involves auxiliary variables and inequalities. This reformulation yields

$$
\begin{aligned}
\min \quad & Z_0 \cdot \theta_0 + \eta t_0 \\
\text{s.t.} \quad & -t_0 \leq \quad \bar{Z}_0 \cdot \bar{\theta}_0 \quad \leq t_0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) + \eta t_n \quad = \quad -\beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& -t_n \leq \; \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) \; \leq t_n \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \sum_{n \in \mathcal{N}_T} p_n x_n - \lambda \sqrt{\sum_{n \in \mathcal{N}_T} p_n (x_n - \sum_{n \in \mathcal{N}_T} p_n x_n)^2} \quad \geq \quad 0 \\
& v_n + x_n \quad = \quad Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\
& v_n \quad \geq \quad 0 \;\forall\, n \in \mathcal{N}_T, \\
& t_n \quad \geq \quad 0 \;\forall\, n \in \mathcal{N}_t, t \geq 1
\end{aligned}
$$

for WSPTC, and

$$
\begin{array}{llll}
\max & -Z_0 \cdot \theta_0 - \eta t_0 & & \\
\text{s.t.} & -t_0 \leq & \bar{Z}_0 \cdot \bar{\theta}_0 & \leq t_0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) + \eta t_n & = & \beta_n F_n, \ \forall \, n \in \mathcal{N}_t, t \geq 1 \\
& -t_n \leq & \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) & \leq t_n \forall \, n \in \mathcal{N}_t, t \geq 1 \\
& \sum_{n \in \mathcal{N}_T} p_n x_n - \lambda \sqrt{\sum_{n \in \mathcal{N}_T} p_n (x_n - \sum_{n \in \mathcal{N}_T} p_n x_n)^2} & \geq & 0 \\
& v_n + x_n & = & Z_n \cdot \theta_n \forall \, n \in \mathcal{N}_T, \\
& v_n & \geq & 0 \, \forall \, n \in \mathcal{N}_T, \\
& t_n & \geq & 0 \, \forall \, n \in \mathcal{N}_t, t \geq 1
\end{array}
$$

for BSPTC.

Using Lagrange duality, we obtain the following dual problems to WSPTC and BSPTC, respectively over variables $q_n, n \in \mathcal{N}_t, 0 \leq t \leq T,\ \alpha_n, \gamma_n \in \mathbb{R}^m, n \in \mathcal{N}_t, 0 \leq t \leq T-1$.

$$\max(\min) \sum_{n>0} \beta_n q_n F_n \tag{11}$$

subject to

$$\|\omega\|_2 \leq \lambda, \tag{12}$$

$$q_n = p_n(1 + \tilde{p}^T \omega) - \sqrt{p_n}\omega_n, \forall n \in \mathcal{N}_T, \tag{13}$$

$$q_m = \sum_{n \in \mathcal{S}(m)} q_n, \ \forall \, m \in \mathcal{N}_t, 0 \leq t \leq T-1 \tag{14}$$

$$q_m \bar{Z}_m - \alpha_m + \gamma_m = \sum_{n \in \mathcal{S}(m)} q_n \bar{Z}_n, \ \forall \, m \in \mathcal{N}_t, 0 \leq t \leq T-1 \tag{15}$$

$$\alpha_m + \gamma_m \leq \eta \bar{Z}_m \ \forall m \in \mathcal{N}_t, 0 \leq t \leq T-1 \tag{16}$$

$$\alpha_m, \gamma_m \geq 0 \ \forall m \in \mathcal{N}_t, 0 \leq t \leq T-1 \tag{17}$$

$$q_0 = 1, \tag{18}$$

$$q_n \geq 0, \ \forall n \in \mathcal{N}_T. \tag{19}$$

After elimination of the ω_n variables as in the previous section, we obtain that the dual problems consist in maximizing (or, minimizing) the function $\sum_{n>0} \beta_n q_n F_n$ over all measures $q_n, n \in \mathcal{N}_T$ (and $q_n \geq 0, \forall n \in \mathcal{N}_t, 0 \leq t \leq T-1$) and $(\alpha, \gamma) = (\alpha_n, \gamma_n \in \mathbb{R}^m_+, n \in \mathcal{N}_t, 0 \leq t \leq T-1)$ satisfying (14)–(15)–(16)–(17)–(9)– (10). Let us denote the feasible set of the above problems by $\mathcal{Q}_\eta(\lambda)$. Then we have the following result.

Theorem 3 *Assuming that problems BSPTR and WSPTR possess strictly feasible solutions we have*

1. *the minimum price charged by a writer of contingent claim F desiring an arbitrage-adjusted Sharpe ratio of terminal wealth positions equal to λ or higher in the presence of proportional transaction costs for trades in the risky assets is given by*

$$\frac{1}{\beta_0} \max_{q,\alpha,\gamma \in \mathcal{Q}_\eta(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t],$$

2. *the maximum price acceptable to the buyer seeking at least an arbitrage-adjusted Sharpe ratio of λ from the terminal wealth positions in presence of proportional transaction costs for trades in the risky assets is given by*

$$\frac{1}{\beta_0} \min_{q,\alpha,\gamma \in \mathcal{Q}_\eta(\lambda)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t].$$

Notice that $\mathcal{Q}(\lambda) \subseteq \mathcal{Q}_\eta(\lambda)$. Hence, we expect the price bounds to widen in the presence of transactions costs.

To find the limiting value of λ we solve the problem MLTR below as a strictly convex quadratic programming problem as in the previous section:

$$\min_{q,\alpha,\gamma} \{\sigma(q./p) : (14) - (15) - (16) - (17) - (10)\}.$$

Example 5. Consider the financial market of Example 1, and the European Call option of Examples 2 and 3. Assume that a 5% transaction cost is incurred for all trades in the stock, i.e., $\eta = 0.05$. The pricing bounds obtained by solving the problems to BA and WA in the presence of transaction costs are 1.6 and 2.64. Compared to the pricing bounds of 2 and 2.2 of Example 2, the above figures give a wider interval as expected. For some λ values, the pricing bounds from the problems WSPTR and

λ	**BSPTR**	**WSPTR**
0.8	2.069	2.564
0.78	2.159	2.551
0.77	2.204	2.544
0.75	2.293	2.527
0.72	2.428	2.489

Table 2: Arbitrage-adjusted Sharpe-ratio-based price bounds for Example 5.

BSPTR are given in Table 2. At the value of $\lambda = 0.7137$, the two bounds coincide at the price 2.463 attained at the measure $q_1 = 0.063$, $q_2 = 0.295$, and $q_3 = 0.642$.

5 Pricing in the Presence of Several Measures

In the previous two sections, our results showed that the choice of original probability measure P plays a role in our asset pricing procedures, whereas in classical, arbitrage-based theory the measure P does not appear except for the fact that the martingale measure used in valuation is drawn from the closure of the set of martingale measures equivalent to the original measure [17]. The dependence of pricing on the original measure, which goes somewhat against the usual practice in financial economics, can be alleviated to some extent if we allow the specification of several original measures in the formulation of pricing problems. This approach was advocated (although the motivation was completely different) in [5] for developing a valuation theory in incomplete markets.

Let P^i, $i = 1, \ldots, \ell$ be a collection of candidate original measures defined on the leaves of the scenario tree describing the financial market, which we will refer to as "trial measures" after [5]. We enforce the arbitrage-adjusted Sharpe ratio criterion using all these measures and based on the belief that they can equally well represent probabilities of the outcomes. We use the following pricing problem for deciding the writer's policy in financing a contingent claim F (referred to as WSPMP)

$$\begin{aligned}
\min \quad & Z_0 \cdot \theta_0 \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) \;=\; -\beta_n F_n,\; \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \sum_{n\in\mathcal{N}_T} p_n^i x_n - \lambda \sqrt{\sum_{n\in\mathcal{N}_T} p_n^i (x_n - \sum_{n\in\mathcal{N}_T} p_n^i x_n)^2} \;\geq\; 0, \forall i = 1,\ldots,\ell, \\
& v_n + x_n \;=\; Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\
& v_n \;\geq\; 0 \;\forall\, n \in \mathcal{N}_T,
\end{aligned}$$

while the buyer's problem of finding the maximum acceptable price is now (BSPMP)

$$\begin{aligned}
\max \quad & -Z_0 \cdot \theta_0 \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) \;=\; \beta_n F_n,\; \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \sum_{n\in\mathcal{N}_T} p_n^i x_n - \lambda \sqrt{\sum_{n\in\mathcal{N}_T} p_n^i (x_n - \sum_{n\in\mathcal{N}_T} p_n^i x_n)^2} \;\geq\; 0, \forall i = 1,\ldots,\ell, \\
& v_n + x_n \;=\; Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\
& v_n \;\geq\; 0 \;\forall\, n \in \mathcal{N}_T.
\end{aligned}$$

Let us denote the variance with respect to measure P^i as σ^{p^i}. The dual problem to WSPMP is the following optimization problem in variables $q^i \in \mathbb{R}_+^N$, $i = 1,\ldots,\ell$, and $\xi \in \mathbb{R}_+^\ell$ and $\omega^i \in \mathbb{R}_+^N$, $i = 1,\ldots,\ell$.

$$\max \sum_{i=1}^{\ell} \sum_{n>0} \beta_n q_n^i F_n \tag{20}$$

subject to

$$\|\omega^{\mathbf{i}}\|_2 \leq \lambda \;\forall i = 1,\ldots,\ell, \tag{21}$$

$$q_n = \sum_{i=1}^{n} p_n^i (\xi_i + \tilde{p}^{iT} \omega^{\mathbf{i}}) - \sqrt{p_n^i}\omega_n^i, \forall n \in \mathcal{N}_T, \tag{22}$$

$$q_m Z_m = \sum_{n\in\mathcal{S}(m)} q_n Z_n,\; \forall\, m \in \mathcal{N}_t, 0 \leq t \leq T-1 \tag{23}$$

$$q_0 = 1, \tag{24}$$

$$q_n \geq 0,\; \forall n \in \mathcal{N}_T, \tag{25}$$

where $\tilde{p}^i$ is the vector with components $\sqrt{p_n^i}$. Solving for ω^i we obtain:

$$\omega_i^n = \xi_i \sqrt{p_n^i}(q_n/p_n^i - 1), \forall n \in \mathcal{N}_T, \forall i = 1, \dots, \ell.$$

Defining $y_n^i = \xi_i q_n$ for all n and $i = 1, \dots, \ell$, and making ℓ copies of the constraints (22) we finally get the equivalent dual

$$\begin{aligned}
\max \quad & \sum_{i=1}^{\ell} \sum_{n>0} \beta_n y_n^i F_n \\
\text{s.t.} \quad & \xi_i Z_0 = \sum_{n \in \mathcal{S}(0)} y_n^i Z_n, \ \forall i = 1, \dots, \ell \\
& y_m^i Z_m = \sum_{n \in \mathcal{S}(m)} y_n^i Z_n, \ \forall\, m \in \mathcal{N}_t, 1 \le t \le T-1, i = 1, \dots, \ell \\
& \sqrt{\sigma^{p^i}(y^i./p^i)} \le \lambda \xi_i, \forall i = 1, \dots, \ell \\
& \mathbb{E}^{P^i}(y^i./p^i) = \xi_i, \forall i = 1, \dots, \ell \\
& \sum_{i=1}^{\ell} \xi_i = 1
\end{aligned}$$

whereas the dual problem to BSPMP is, as usual, the optimization problem with identical objective function and constraints, except that maximization is replaced with minimization. The derivation of the Lagrange dual problem is similar to our earlier derivation given in the Appendix, hence omitted. It can be noted that we did not express the objective function as an expectation since the vectors y^i are not, strictly speaking, probability measures. This follows from the observation that for any fixed i, $i = 1, \dots, \ell$, the y^i values corresponding to the leaf nodes add up to ξ_i. However, we could still use an expectation in the objective function due to the following reasoning. In the primal problem WSPMP (or, BSPMP) we are enforcing the arbitrage-adjusted Sharpe-ratio bound for several measures. This set of constraints is equivalent to maximizing a convex left hand side function over a discrete set, where the maximum is attained for at least one of the trial measures in the discrete set. Hence, in the dual problem, provided that it is solvable, there is always an optimal solution with a single ξ_i equal to 1 while all others are equal to zero, by a complementarity argument. Hence, the corresponding y^i will act as a measure, and the use of an expectation is warranted.

Denote the common feasible set of the dual problems $\mathcal{QP}(\lambda)$. Then we have the analog of Theorem

2 and Theorem 3.

Theorem 4 *Assuming that problems BSPMP and WSPMP possess strictly feasible solutions we have*

1. *the minimum price charged by a writer of contingent claim F desiring an arbitrage-adjusted Sharpe ratio of terminal wealth positions equal to λ or higher in the presence of trial measures $P^i, i = 1, \ldots, \ell$ is given by*

$$\frac{1}{\beta_0} \max_{y^i, i=1,\ldots,\ell, \xi \in \mathcal{QP}(\lambda)} \sum_{i=1}^{\ell} \mathbb{E}^{Y^i}[\sum_{t=1}^{T} \beta_t F_t],$$

2. *the maximum price acceptable to the buyer seeking at least an arbitrage-adjusted Sharpe ratio of λ from the terminal wealth positions in the presence of trial measures $P^i, i = 1, \ldots, \ell$ is given by*

$$\frac{1}{\beta_0} \min_{y^i, i=1,\ldots,\ell, \xi \in \mathcal{QP}(\lambda)} \sum_{i=1}^{\ell} \mathbb{E}^{Y^i}[\sum_{t=1}^{T} \beta_t F_t].$$

The smallest value of λ for which the dual feasible set $\mathcal{QP}(\lambda)$ is non-empty is obtained by solving the problem:

$$\begin{array}{lrcl} \max & \lambda & & \\ \text{s.t.} & \xi_i Z_0 & = & \sum_{n \in \mathcal{S}(0)} y_n^i Z_n, \ \forall i = 1, \ldots, \ell, \\ & y_m^i Z_m & = & \sum_{n \in \mathcal{S}(m)} y_n^i Z_n, \ \forall \, m \in \mathcal{N}_t, 0 \le t \le T-1, i = 1, \ldots, \ell \\ & \sqrt{\sigma^{p^i}(y^i./p^i)} & \le & \lambda \xi_i, \forall i = 1, \ldots, \ell \\ & \mathbb{E}^{P^i}(y^i./p^i) & = & \xi_i, \forall i = 1, \ldots, \ell \\ & \sum_{i=1}^{\ell} \xi_i & = & 1 \end{array}$$

over the variables λ, $y^i \in \mathbb{R}_+^N$, $i = 1, \ldots, \ell$, and $\xi \in \mathbb{R}_+^\ell$. It is immediate to notice that the above problem is a non-convex optimization problem as it involves the products of variables λ and ξ_i. However, the following convex reformulation obtained by defining the variables a_i, for $i = 1, \ldots, \ell$

and associated constraints is equivalent to the above non-convex formulation:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{\ell} a_i \\
\text{s.t.} \quad & \xi_i Z_0 = \sum_{n \in \mathcal{S}(0)} y_n^i Z_n, \ \forall i = 1, \dots, \ell, \\
& y_m^i Z_m = \sum_{n \in \mathcal{S}(m)} y_n^i Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \le t \le T-1, i = 1, \dots, \ell \\
& \sqrt{\sigma^{p^i}(y^i./p^i)} \le a_i, \forall i = 1, \dots, \ell \\
& \mathbb{E}^{P^i}(y^i./p^i) = \xi_i, \forall i = 1, \dots, \ell \\
& \sum_{i=1}^{\ell} \xi_i = 1 \\
& a_i \le M\xi_i, \forall i = 1, \dots, \ell
\end{aligned}
$$

where M is a suitable positive constant. In our computational experience, $M = 10$ seems to be a fine choice. Notice that the existence of an optimal solution (provided the problem is solvable) alluded to above, where exactly one of the ξ_is is equal to one, forces the corresponding a_i to assume the value one.

Example 6. Let us consider the financial market and European call option with strike $K = 9$ from Example 1 with the trial measures $P^1 = (1/3, 1/3, 1/3)$ (the original measure used in our previous examples), $P^2 = (1/6, 1/6, 2/3)$ and $P^3 = (1/6, 2/3, 1/6)$. The buyer and writer pricing bounds for different values of λ are displayed in Table 3. At the value of $\lambda = 0.1690309$, the two bounds coincide at the price 2.114 attained at the martingale measure $q_1 = 0.114$, $q_2 = 0.143$, and $q_3 = 0.743$. In this particular example, it is measure P^2 that dominates the other two measures. Compared to the common price of 2.026 of Example 3 obtained using measure P^1 only, the current price is higher. This upward shift in the equilibrium price seems to be caused by a higher probability mass concentration on the outcome corresponding to node 3 where the price of the stock drops to 7.5.

We close this section by a remark concerning the representation of the set of trial measures. A

λ	**BSPMP**	**WSPMP**
0.21	2.088	2.14
0.2	2.092	2.136
0.19	2.096	2.132
0.18	2.101	2.127
0.17	2.11	2.118

Table 3: Arbitrage-adjusted Sharpe-ratio-based price bounds for Example 6.

general formulation GWSPMP for WSPMP can be expressed as follows:

$$
\begin{aligned}
\min \quad & Z_0 \cdot \theta_0 \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \lambda \sqrt{\sum_{n \in \mathcal{N}_T} p_n (x_n - \sum_{n \in \mathcal{N}_T} p_n x_n)^2} \leq \sum_{n \in \mathcal{N}_T} p_n x_n, \forall p \in \mathcal{P}, \\
& v_n + x_n = Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\
& v_n \geq 0 \,\forall\, n \in \mathcal{N}_T,
\end{aligned}
$$

where $\mathcal{P}$ is the set of trial measures. The set $\mathcal{P}$ in the present section was chosen to be a discrete set. However, this set could very well be a polyhedral set or an ellipsoidal set [1]. When the uncertain parameters on the two sides of a second-order cone inequality are subject to independent uncertainty, one can find convex equivalent representations of the uncertain conic inequality [2]. Unfortunately, in our case above finding a compact convex reformulation of GWSPMP is a difficult problem (see e.g., [2]) since the measure P that we make subject to variation occurs on both sides of the inequality. Nonetheless, approximate convex reformulations which can be efficiently solved exist for this case. These approximations are not the subject of the present paper, and will be developed elsewhere.

6 Conclusions

We studied the problem of pricing and hedging contingent claims in incomplete markets in a multi-period convex optimization (discrete-time, finite probability space) framework. We developed a pricing concept based on the arbitrage-adjusted Sharpe ratio of final wealth positions, which allow to obtain arbitrage only in the limit as a risk aversion parameter tends to infinity, in the context of discrete time discrete state investment problems. We analyzed the resulting optimization problems using convex programming duality. All optimization models developed in the paper can be solved by off-the-shelf optimization software. We showed that the pricing bounds obtained from our analysis are tighter than the no-arbitrage pricing bounds. Furthermore, our results indicate that the writer and buyer prices of a contingent claim can coincide for a limiting value of the risk aversion parameter imposed by the financial market. We extended the pricing model to include proportional transaction costs in the trades of risky assets. To render the pricing results less dependent on the original probability measure governing the market, we proposed a pricing framework allowing the specification of multiple trial measures.

A Appendix: Derivation of Dual Problem for WSP

The writer's problem for financing a contingent claim F that we referred to as WSP is rewritten as follows for convenience.

$$\begin{array}{lrcl}
\min & Z_0 \cdot \theta_0 & & \\
\text{s.t.} & Z_n \cdot (\theta_n - \theta_{\pi(n)}) & = & -\beta_n F_n, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \sum_{n \in \mathcal{N}_T} p_n x_n & = & t \\
& t - \lambda \sqrt{\sum_{n \in \mathcal{N}_T} p_n (x_n - t)^2} & \geq & 0 \\
& v_n + x_n & = & Z_n \cdot \theta_n \forall\, n \in \mathcal{N}_T, \\
& v_n & \geq & 0 \; \forall\, n \in \mathcal{N}_T.
\end{array}$$

The above problem has the following Lagrange dual problem after viewing

$$\lambda\sqrt{\sum_{n\in\mathcal{N}_T} p_n(x_n - \sum_{n\in\mathcal{N}_T} p_n x_n)^2}$$

as the two-norm of the vector with $|\mathcal{N}_T|$ components given by $\sqrt{p_n}(x_n - \sum_{n\in\mathcal{N}_T} p_n x_n)$ and using the variational representation of the two-norm:

$$\max_{\|\omega\|_2\le\lambda, q, w, V}\ \min_{\Theta, x, v\ge 0, t} L(\Theta, V, x, v, q, w, \xi, \omega, t)$$

where

$$\begin{aligned} L(\Theta, V, x, v, q, \xi, \omega, t) \quad = \quad & Z_0\cdot\theta_0 + \sum_{t=1}^{T}\sum_{n\in\mathcal{N}_t} q_n\left(Z_n\cdot(\theta_n - \theta_{\pi(n)}) + \beta_n F_n\right) + \sum_{n\in\mathcal{N}_T} w_n(Z_n\cdot\theta_n - v_n - x_n) + \\ & \xi\left(\sum_{n\in\mathcal{N}_T}\omega_n\sqrt{p_n}(x_n - t)\right) + V(t - \sum_{n\in\mathcal{N}_T} p_n x_n). \end{aligned}$$

Minimizing with respect to θ_0 we obtain

$$q_m Z_m = \sum_{n\in\mathcal{S}(m)} q_n Z_n,\ \forall\ m\in\mathcal{N}_t, 0\le t\le T-1$$

which are equations (5). Minimization with respect to θ_n, for $n\in\mathcal{N}_t, t\ge 1$ yields $q_n = -w_n$, for $n\in\mathcal{N}_T$. Minimization with respect to $v_n\ge 0$ gives $w_n\le 0$, for $n\in\mathcal{N}_T$. Therefore, we have $q_n\ge 0$ for all $n\in\mathcal{N}_T$. Minimization with respect to x_n yields

$$-w_n + \xi\omega_n\sqrt{p_n} - Vp_n = 0$$

for all $n\in\mathcal{N}_T$. Replacing $-w_n$ by q_n we obtain

$$q_n + \xi\omega_n\sqrt{p_n} - Vp_n = 0$$

for all $n\in\mathcal{N}_T$. Finally, minimizing over t we obtain

$$V = \xi(\tilde{p}^T\omega - 1).$$

Substituting this expression for V in the equations

$$q_n + \xi\omega_n\sqrt{p_n} - Vp_n = 0$$

for all $n\in\mathcal{N}_T$ we obtain using the facts $\sum_{n\in\mathcal{N}_T} q_n = 1$ and $\sum_{n\in\mathcal{N}_T} q_n = 1$ that $\xi = 1$. Therefore, we can replace all occurrences $\xi\omega_n$ by ω_n.

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